Optimisation, contrôle, données

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Responsable: Dario Prandi, dario.prandi@centralesupelec.fr
Chargé de TD: Kevin Le Balc'h, kevin.le-balc-h@inria.fr

Exercise 1: Primal/Dual problems

We consider the following optimization problem (OP)

$$\min_{(x,y)\in\mathbb{R}^2} (x-1)^2 + (y-2)^2,$$

subject to the following constraints

$$x \leq y,\ \ x + 2y \leq 2.$$

1. Why does (OP) admit a solution?
2. Write the Lagrange function associated to (OP).
3. Solve (OP) by using KKT's theorem.
4. Write the dual Lagrange function.
5. Do we have strong duality?

Exercise 2: Proximal operator and Moreau regularization

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. We define the *proximal operator* of f as

$$\operatorname{prox}_f : x \mapsto \arg\min_{y\in\mathbb{R}^n} \left\{ f(y) + \frac{1}{2}\|y-x\|^2 \right\},$$

and the *Moreau regularization* of parameter $\mu > 0$ as

$$f_\mu : x \mapsto \inf_{y\in\mathbb{R}^n} \left\{ f(y) + \frac{1}{2\mu}\|y-x\|^2 \right\}.$$

We consider the proximal point algorithm defined by the sequence

$$x^{(k+1)} = \operatorname{prox}_{\mu f}(x^{(k)}).$$

1. Show that prox_f and f_μ are well-defined. For C closed convex and non-empty, and $f = \chi_C$ (the 0-∞ characteristic function of C), identify prox_f and f_μ.
2. Show that $x^\star$ is a minimizer of f if and only if it minimizes f_μ, if and only if $x^\star = \operatorname{prox}_f(x^\star)$.
3. Show that
$$f_\mu(x) = \frac{1}{2\mu}\|x\|^2 - \frac{1}{\mu}\left(\mu f + \frac{1}{2}\|\cdot\|^2\right)^*(x),$$
where $(\cdot)^*$ denotes the Fenchel conjugate.
4. Show that
$$\operatorname{prox}_{\mu f}(x) = \arg\max_y \left\{ x^\top y - \mu f(y) - \frac{1}{2}\|y\|^2 \right\}.$$
5. Show that
$$\nabla f_\mu(x) = \frac{1}{\mu}\left(x - \operatorname{prox}_{\mu f}(x)\right).$$
and interpret the proximal point algorithm as a gradient algorithm.
Hint: Recall that $v \in \partial g(x)$ if and only if $v \in \arg\max_y v^\top y - g(y)$, if and only if $x \in \partial g^*(v)$.

6. Writing
$$f_\mu(x) = \min_{y,z} \left[f(y) + \frac{1}{2\mu}\|z\|^2 \right] \qquad \text{s.t.} \qquad x - y = z,$$
and using duality, show that
$$f_\mu(x) = \left(f^* + \frac{\mu}{2}\|\cdot\|^2 \right)^* (x).$$
This implies that f_μ has a $\frac{1}{\mu}$-Lipschitz gradient.
7. Show that if f is a proper convex lower-semicontinuous function admitting a minimizer, the proximal point algorithm converges toward a minimizer of f.

Exercise 3: LQ optimal control problem

Consider the two 2-dimensional linear systems driven by the same scalar input $u(t)$:
$$\dot{x}_1 = A_1 x_1 + B_1 u, \qquad A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$\dot{x}_2 = A_2 x_2 + B_2 u, \qquad A_2 = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$
Let the augmented state be $X = [x_1^\top, x_2^\top]^\top \in \mathbb{R}^4$, and define
$$\mathcal{A} = \text{diag}(A_1, A_2), \qquad \mathcal{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

1. For each pair (A_i, B_i), determine whether the system is controllable.
2. Determine whether the two systems are *simultaneously controllable* by the single input u and, if it is not, describe the non-reachable states.
3. Replace A_2 by the parametric matrix
$$A_2(\alpha) = \begin{bmatrix} -2 & \alpha \\ 1 & 0 \end{bmatrix}.$$
For which $\alpha \in \mathbb{R}$ is the augmented pair $(\mathcal{A}(\alpha), \mathcal{B})$ controllable? Provide the condition(s) discuss their relationship to the controllability of each of the systems separately.

Consider the LQ problem for the augmented system
$$\dot{X} = \mathcal{A}X + \mathcal{B}u, \qquad X(0) = X_0,$$
with final time $T > 0$ and cost
$$J(u) = \tfrac{1}{2} \int_0^T \left(X(t)^\top R X(t) + a\, u(t)^2 \right) dt,$$
where $R = \text{diag}(R_1, R_2)$, each $R_i \in \mathbb{R}^{2\times 2}$ is symmetric positive semidefinite, and $a > 0$.

4. Write the expression for the optimal control of this unconstrained terminal state problem in terms of the matrix $E(t)$, which satisfies the Riccati differential equation
$$\dot{E} = R - \mathcal{A}^\top E - E\mathcal{A} - \frac{1}{r} E\mathcal{B}\mathcal{B}^\top E,$$
with appropriate boundary conditions at $E(T)$, to be determined. Write the Riccati equation explicitly using the matrices $\mathcal{A} = \text{diag}(A_1, A_2)$, $\mathcal{B}$, and R, and the block decomposition of E:
$$E = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}.$$

5. Discuss why the optimal control u^* exists and is unique for any initial state $X(0)$, even if the augmented system $(\mathcal{A}, \mathcal{B})$ is not controllable.
6. Is it possible for the optimal control to act independently on the the two subsystems? *Hint: How this would reflect on the Riccati equation solution?*

Exercise 4: Bilinear control problem and linear ResNets

Consider the following control problem:

$$\dot{x} = Ux, \qquad x \in \mathbb{R}^d,\ U \in \mathbb{R}^{d\times d}. \tag{1}$$

1. We aim at giving necessary and sufficient conditions for the existence of a time-independent control $U \in L^\infty([0,1], \mathbb{R}^{d\times d})$ steering the system from $x_0 \in \mathbb{R}^d$ to $y_0 \in \mathbb{R}^d$.
   (a) Recall that the solution of (1) is $x(t) = R(t)x(0)$, where $R(t) = R(t,0)$ is the state-transition matrix $R(t)$, which satisfies

   $$\frac{\partial}{\partial t} R(t) = U(t)R(t), \qquad R(0) = \mathrm{Id}\,.$$

   Use this fact to reduce controllability of (1) to a controllability problem for the state-transition matrix to $R(1)$ such that $y_0 = R(1)x_0$. Conclude for the case $x_0 = 0$ and $y_0 \neq 0$.

   (b) For the case $x_0 \neq 0$ and y_0 linearly independent of x_0, show that there exists $v \perp (y_0 - x_0)$ and $\eta \in \mathbb{R}$ such that $U = \eta(y_0 - x_0)v^\top$ steers the system from x_0 to y_0. *Hint: compute* $R(1) = \exp(U)$ *using the series expansion of the matrix exponential.*

   (c) Consider $d = 2$ and show that in general it is sufficient to consider a control of the form

   $$U = \gamma J + \eta\,\mathrm{Id}, \qquad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \gamma, \eta \in \mathbb{R}.$$

   Deduce the form of the control for general $d \geq 2$.
   Hint: Recall that $\exp(\gamma J)$ *is the counter-clockwise rotation matrix of angle* γ *and that* $e^{A+B} = e^A e^B$ *if the matrices* A *and* B *commute.*

2. Consider the following simultaneous controllability problem: Given two sets $\{x_1, \ldots, x_d\}$ and $\{y_1, \ldots, y_d\}$ which are bases of $\mathbb{R}^d$ find a *single* time-independent control U steering the system from each x_i to y_i, $i = 1, \ldots, d$.
   (a) Show that the this problem is controllable with time-independent controls if and only if the map $U \in \mathbb{R}^{d\times d} \mapsto \exp U$ is surjective onto the set of invertible matrices.
   (b) Show that this is false in general.
   (c) For $d = 2$, let $X = [x_1, x_2]$ and $Y = [y_1, y_2]$. Show that a sufficient condition for simultaneous controllability is that $X^{-1}Y$ has positive determinant and admits two distinct eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C} \setminus (-\infty, 0)$.

3. Write down the ResNet with L layers corresponding to the neural ODE of equation (1). What can we say about the universal approximating properties of this family of ResNets? *Hint: Are there general restriction on simultaneous controllability with time-dependent controls?*