Optimisation, contrôle, données

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Exercise 1: Constrained optimization and Fenchel duality

We consider the optimization problem (P)

$$\min_{(x,y)\in\mathbb{R}^2} x^2 + 2y^2 - 2x - 4y,$$

subject to the constraints

$$x + y \le 1, \qquad x - y \le 0.$$

1. Show that (P) admits a unique solution.
2. Write the Lagrangian associated with (P) and state the KKT conditions.
3. Verify that Slater's condition holds. Solve (P) using KKT's theorem.
4. Write the dual problem and verify strong duality.

Exercise 2: Projected gradient descent and convergence

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function of class C^1 with L-Lipschitz gradient, and let $C \subset \mathbb{R}^n$ be a non-empty, closed, convex set. We consider the *projected gradient descent* algorithm defined by

$$x^{(k+1)} = \Pi_C\left(x^{(k)} - \alpha\nabla f(x^{(k)})\right),$$

where Π_C denotes the orthogonal projection onto C and $\alpha > 0$ is the step size.

1. Recall why Π_C is well-defined on $\mathbb{R}^n$. Show that, for any $x \in \mathbb{R}^n$ and $y \in C$,

   $$\langle x - \Pi_C(x),\, y - \Pi_C(x)\rangle \le 0.$$

   Hint: for the second part, letting $p = \Pi_C(x)$, justify and use the fact that $\|x-p\|^2 \le \|x-z\|^2$ for $z = p + t(y-p)$, $t \ge 0$.

2. Show that Π_C is 1-Lipschitz, i.e., for all $x, z \in \mathbb{R}^n$,

   $$\|\Pi_C(x) - \Pi_C(z)\| \le \|x - z\|.$$

3. Show that $x^\star \in C$ is a minimizer of f on C if and only if, for any $\alpha > 0$,

   $$x^\star = \Pi_C(x^\star - \alpha\nabla f(x^\star)).$$

   Hint: reformulate the problem as an unconstrained one, using χ_C, the $0-\infty$ characteristic function of C. Apply the minimization criterion for the subdifferential and use question 1.

4. Show that, for any $x^\star$ minimizer of f over C and for any $k \ge 0$,

   $$\|x^{(k+1)} - x^\star\|^2 \le \|x^{(k)} - x^\star\|^2 - \alpha\left(\frac{2}{L} - \alpha\right)\|\nabla f(x^{(k)}) - \nabla f(x^\star)\|^2 - 2\alpha\left(f(x^{(k)}) - f(x^\star)\right).$$

   Hint: use the characterization of the projection from question 1, and the quadratic upper bound

   $$f(y) \le f(x) + \langle\nabla f(x), y - x\rangle + \frac{L}{2}\|y - x\|^2, \qquad \forall x, y \in \mathbb{R}^n$$

   .

5. Show that for $0 < \alpha < 2/L$, the sequence $\{f(x^{(k)})\}$ is non-increasing.

6. Deduce that for $0 < \alpha < 2/L$, the sequence $\{f(x^{(k)})\}$ converges to $f(x^\star)$ and

$$f(x^{(k)}) - f(x^\star) \le \frac{\|x^{(0)} - x^\star\|^2}{2\alpha k}.$$

7. Now consider the specific problem of minimizing $f(x) = \frac{1}{2}\|Ax - b\|^2$ over $C = \{x \in \mathbb{R}^n : x \ge 0\}$ (non-negative least squares), where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.
   (a) Write ∇f and determine the Lipschitz constant L of the gradient.
   (b) Write explicitly the projected gradient iteration for this problem.

Exercise 3: Controllability and LQ optimal control

Consider the linear control system in $\mathbb{R}^n$:

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0,$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $u \in L^\infty([0,T], \mathbb{R}^m)$. Consider the LQ optimal control problem on $[0,T]$: minimize

$$J(u) = \frac{1}{2}\int_0^T \left(x(t)^\top Q\, x(t) + u(t)^\top R\, u(t)\right) dt + \frac{1}{2}\, x(T)^\top S\, x(T), \tag{LQ-cost}$$

where $Q, S \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite and $R \in \mathbb{R}^{m\times m}$ is symmetric positive definite.

1. Consider the following system:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

   (a) Determine whether the system is controllable, and describe the reachable set from a given initial state x_0.
   (b) State the matrix Riccati equation satisfied by $E(t)$ and give the expression for the optimal control $u^\star(t)$ in feedback form.
   (c) Does the optimal control exist and is it unique, even if the system (A, B) is not controllable? Justify.

2. We now consider the *double integrator* ($n = 2$, $m = 1$):

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad Q = I_2, \qquad R = 1.$$

   (a) Verify that (A, B) is controllable.
   (b) Let $E_T(t)$ denote the solution of the finite-horizon Riccati equation for the cost functional J of Equation (LQ-cost) with $S = 0$. Assume that, for fixed t, $E_T(t)$ converges to a limit matrix $E_\infty \succeq 0$ as $T \to \infty$. Show that E_∞ satisfies the algebraic Riccati equation:
   $$Q + A^\top E_\infty + E_\infty A - E_\infty B R^{-1} B^\top E_\infty = 0.$$
   (c) Writing $E_\infty = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, expand the algebraic Riccati equation into a system of scalar equations for a, b, c, d and solve it explicitly.
   (d) Give the optimal feedback control $u^\star = K^\star x$ corresponding to E_∞ and write the closed-loop matrix $A_{\text{cl}} = A - BK^\star$. Verify that A_{cl} is Hurwitz (i.e., is such that the closed-loop system $\dot{x} = A_{\text{cl}} x$ is asymptotically stable).

Exercise 4: Neural ODEs and ResNets

Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a nonlinear activation function applied component-wise. Let $\theta : [0,1] \to \mathbb{R}^p$ be a time-varying parameter and consider the *neural ODE*

$$\dot{x} = f_\theta(t,x) = W(t)\,\sigma(V(t)\,x + c(t)), \qquad x(0) = x_0 \in \mathbb{R}^n, \tag{1}$$

where $W(t) \in \mathbb{R}^{n\times q}$, $V(t) \in \mathbb{R}^{q\times n}$, $c(t) \in \mathbb{R}^q$, and $\theta(t) = (W(t), V(t), c(t))$ are learnable parameters.

1. Write down the ResNet with L layers corresponding to the Euler discretization of (1) on $[0,1]$ with step $h = 1/L$.
2. Let $\Phi_t : \mathbb{R}^n \to \mathbb{R}^n$ denote the flow map of (1), i.e., $x(t) = \Phi_t(x_0)$. Show that

   $$\det\bigl(D\Phi_t(x_0)\bigr) > 0, \qquad \forall\, t \in [0,1],\ \forall\, x_0 \in \mathbb{R}^n.$$

   What does this imply about the expressiveness of neural ODEs?
   Hint: use the Liouville–Jacobi formula $\frac{d}{dt}\det(D\Phi_t) = \det(D\Phi_t)\ \mathrm{tr}(D_x f_\theta(t,\Phi_t))$.
3. To overcome this limitation, one considers the *augmented neural ODE*:

   $$\frac{d}{dt}\begin{pmatrix} x \\ z \end{pmatrix} = g_\theta(t,x,z), \qquad \begin{pmatrix} x(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \end{pmatrix},$$

   Here, $(x,z) \in \mathbb{R}^{n+n'}$ is the augmented state. The function g_θ has the form (1) but with x replaced by (x,z) and learnable parameters of appropriate dimensions. The output map is then $\Psi : x_0 \mapsto x(1)$ (projecting back onto the first n coordinates).

   (a) Explain why Ψ is no longer constrained to have positive Jacobian determinant.

   (b) Provide an explicit example with $n = n' = 1$ and linear activation function σ, where Ψ realizes the map $x_0 \mapsto -x_0$.
   Hint: consider a simple linear system that rotates the state in the (x,z)*-plane.*