Geometric Control Theory

March 27th, 2026

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Exercise 1: Controllability

Let $SL(2)$ denote the special linear group of 2×2 matrices with determinant equal to 1. That is:

$$SL(2) = \{Q \in \mathbb{R}^{2\times 2} : \det Q = 1\}.$$

1. Show that $SL(2)$ is a manifold of dimension 3.
   Hint: Recall Jacobi's formula for invertible matrices Q ($d\det(Q)(A) = \operatorname{tr}\big(Q^{-1}A\big)$ for any $A \in \mathbb{R}^{2\times 2}$).

2. Show that the tangent space at $Q \in SL(2)$ is given by:

$$T_Q SL(2) = \{QA \in \mathbb{R}^{2\times 2} : \operatorname{tr} A = 0\}.$$

3. Prove that for any couple of vector fields $X_1(Q) = QA_1$ and $X_2(Q) = QA_2$ on $SL(2)$, their Lie bracket is given by

$$[X_1, X_2](Q) = Q[A_1, A_2], \qquad \text{where} \qquad [A_1, A_2] = A_1A_2 - A_2A_1$$

   Hint: The flow of X_i can be expressed using the matrix exponential. This allows us to consider its Taylor expansion with respect to t.

Consider the following control problem on $SL(2)$:

$$\dot{Q}(t) = Q(t)\left(u_1(t)A + u_2(t)B\right), \qquad \text{where} \qquad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

We consider $Q(0) = \mathrm{Id}$, the identity matrix, as initial condition.

4. Show that the system is Lie bracket generating.

5. Is the system controllable if $u = (u_1, u_2)$ is allowed to take values in $\mathbb{R}^2$? Is it small time locally controllable?

6. Assume now that u is allowed to take values in $[0, 1] \times [-2, 2]$.

   (a) Can we apply Chow-Rashevskii's theorem?

   (b) Let us consider $u = (-1, -1 + v)$ where $v \in [-1, 3]$. Is the system controllable with this restricted control set? Is the system small time locally controllable?
   Hint: The flow of the vector field $X_0(Q) = Q(A - B)$ can be explicitly computed using the matrix exponential.

   (c) Deduce controllability of the system with $u \in [0, 1] \times [-2, 2]$.

Exercise 2: Time-optimal control of a particle with friction

Consider a point particle of unit mass moving on the real line, subject to linear friction and a bounded external force. Its state is described by the position $x \in \mathbb{R}$ and the velocity $v \in \mathbb{R}$. The dynamics are:

$$\dot{x}(t) = v(t), \qquad \dot{v}(t) = -v(t) + u(t), \qquad u(t) \in [-1, 1].$$

1. Write the system in the standard form $\dot{q} = f_0(q) + u\, f_1(q)$ with $q = (x, v)^\top$. Identify the drift f_0 and the controlled vector field f_1.

2. Using the Kalman rank condition, show that the system is controllable.

3. We wish to steer the system from a given initial state (x_0, v_0) to the origin $(0, 0)$ in minimum time. Prove the existence of a minimal time trajectory.

4. Write the pre-Hamiltonian of the Pontryagin Maximum Principle for this problem, and determine the structure of the regular extremal controls.

5. Write the Hamiltonian equations and solve them explicitly. Show that extremal trajectories are *bang-bang* with at most one switching.
Hint: Express v as a function of x.

6. Sketch the optimal synthesis to the origin. What is the switching curve?

Exercise 3: A sub-Riemannian control system: the Grushin plane

We consider the following control system on $\mathbb{R}^2$:

$$\dot{x} = u_1, \qquad \dot{y} = xu_2, \qquad u = (u_1, u_2) \in \mathbb{R}^2.$$

For this system, we aim to solve the sub-Riemannian optimal control problem of steering from $q_{\text{in}} = (0, 0)$ to $q_{\text{fi}} \in \mathbb{R}^2$, with $u \in L^\infty([0, T]; \mathbb{R}^2)$, while minimising the cost

$$E(u) = \int_0^T (u_1^2 + u_2^2)\, dt \to \min .$$

1. Show that the system is controllable and discuss the existence of minimisers to the optimal control problem.

2. Write the Hamiltonian of the PMP and determine the structure of the regular extremal controls.

3. Discuss singular and abnormal extremals.

4. Write the Hamiltonian equations and solve them explicitly. Describe extremal trajectories.
Hint: Justify that the initial covector satisfies $(p_x^0, p_y^0) = (\pm 1, a)$ for $a \in \mathbb{R}$.

5. Show that the trajectory corresponding to $a = 0$ is optimal for all times. Then show that for an initial covector with $a \neq 0$ there exists a time $T_a > 0$ and another covector such whose trajectory arrives at the same final point at time T_a.