Geometric Control Theory

March 27th, 2026

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Exercise 1: Controllability

Let $SL(2)$ denote the special linear group of 2×2 matrices with determinant equal to 1. That is:

$$SL(2) = \{Q \in \mathbb{R}^{2\times 2} : \det Q = 1\}.$$

1. Show that $SL(2)$ is a manifold of dimension 3.
   Hint: Recall Jacobi's formula for invertible matrices Q ($d\det(Q)(A) = \operatorname{tr}\big(Q^{-1}A\big)$ for any $A \in \mathbb{R}^{2\times 2}$).

2. Show that the tangent space at $Q \in SL(2)$ is given by:

$$T_Q SL(2) = \{QA \in \mathbb{R}^{2\times 2} : \operatorname{tr} A = 0\}.$$

3. Prove that for any couple of vector fields $X_1(Q) = QA_1$ and $X_2(Q) = QA_2$ on $SL(2)$, their Lie bracket is given by

$$[X_1, X_2](Q) = Q[A_1, A_2], \qquad \text{where} \qquad [A_1, A_2] = A_1A_2 - A_2A_1$$

   Hint: The flow of X_i can be expressed using the matrix exponential. This allows us to consider its Taylor expansion with respect to t.

Consider the following control problem on $SL(2)$:

$$\dot{Q}(t) = Q(t)\left(u_1(t)A + u_2(t)B\right), \qquad \text{where} \qquad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

We consider $Q(0) = \mathrm{Id}$, the identity matrix, as initial condition.

4. Show that the system is Lie bracket generating.

5. Is the system controllable if $u = (u_1, u_2)$ is allowed to take values in $\mathbb{R}^2$? Is it small time locally controllable?

6. Assume now that u is allowed to take values in $[0, 1] \times [-2, 2]$.
   (a) Can we apply Chow-Rashevskii's theorem?
   (b) Let us consider $u = (-1, -1 + v)$ where $v \in [-1, 3]$. Is the system controllable with this restricted control set? Is the system small time locally controllable?
   Hint: The flow of the vector field $X_0(Q) = Q(A - B)$ can be explicitly computed using the matrix exponential.
   (c) Deduce controllability of the system with $u \in [0, 1] \times [-2, 2]$.

Solution **Question 1.** By definition $SL(2) = \{Q \in \mathbb{R}^{2\times 2} : f(Q) = 1\}$, where $f : \mathbb{R}^{2\times 2} \to \mathbb{R}$ is defined by $f(Q) = \det Q$. Since det is a smooth function, $SL(2)$ is a submanifold of $\mathbb{R}^{2\times 2}$ if 1 is a regular value of f. We compute the differential of f at Q by using Jacobi's formula:

$$df_Q(A) = \operatorname{tr}\big(Q^{-1}A\big).$$

Since Q is invertible, we can choose $A = Q$ to get $df_Q(Q) = \operatorname{tr}(\mathrm{Id}) = 2 \neq 0$. Hence, 1 is a regular value of f, and $SL(2)$ is a submanifold of $\mathbb{R}^{2\times 2}$ of codimension 1. Since $\mathbb{R}^{2\times 2}$ has dimension 4, we conclude that $\dim SL(2) = 3$.

Question 2. Consider $\gamma : (-\varepsilon, \varepsilon) \to SL(2)$ defined by $\gamma(t) = Q + tQA$. We have $\gamma(0) = Q$ and

$\dot\gamma(0) = QA$. We have

$$\det\gamma(t) = 1 \implies 0 = \frac{d}{dt}\Big|_{t=0} \det\gamma(t) = \frac{d}{dt}\Big|_{t=0} \det(Q + tQA) = d\det(Q)(QA) = \operatorname{tr}\big(Q^{-1}QA\big) = \operatorname{tr} A.$$

Hence, for any $QA \in T_Q SL(2)$ it holds $\operatorname{tr} A = 0$. Conversely, if $\operatorname{tr} A = 0$, then $\det(Q + tQA) = 1$ for all t, and hence $QA \in T_Q SL(2)$. We conclude that $T_Q SL(2) = \{QA : \operatorname{tr} A = 0\}$.

Question 3. Let $X_1(Q) = QA_1$ and $X_2(Q) = QA_2$ be the vector fields associated to the control system. By linearity, we have that

$$e^{tX_i}(Q) = Qe^{tA_i} = Q\left(\sum_{k=0}^{\infty} \frac{t^k A_i^k}{k!}\right), \qquad i = 1, 2. \tag{1}$$

Hence, we compute their Lie bracket via the formula:

$$[X_1, X_2](Q) = \frac{1}{2}\frac{d^2}{dt^2}\Big|_{t=0} \varphi(t), \qquad \varphi(t) = e^{-tX_2}e^{-tX_1}e^{tX_2}e^{tX_1}Q = Q\left(e^{-tA_2}e^{-tA_1}e^{tA_2}e^{tA_1}\right).$$

Developing the right hand side via (1) we get that

$$\varphi(t) = Q\left(\operatorname{Id} + t^2(A_1A_2 - A_2A_1) + o(t^2)\right).$$

We conclude that

$$[X_1, X_2](Q) = Q(A_1A_2 - A_2A_1).$$

Question 4. Let $X_A(Q) = QA$ and $X_B(Q) = QB$ be the vector fields associated to the control system. By the previous question, we have that

$$[X_A, X_B](Q) = Q(AB - BA) = QC, \qquad \text{where} \qquad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since A, B and C are linearly independent, we conclude that the system is Lie bracket generating.

Question 5. Since the controls take value in $\mathbb{R}^2$, the system is symmetric. Moreover, since the system is Lie bracket generating, it is controllable by Chow-Rashevskii's theorem. As seen during the course, this theorem actually implies that the system is small time locally controllable.

Question 6a. Since the controls take value in $[0, 1] \times [-2, 2]$, the system is not symmetric (for instance $X_A \in \mathcal{F}$ but $-X_A \notin \mathcal{F}$, since $u_1 = -1$ is not admissible), and thus we cannot apply Chow-Rashevskii's theorem.

Question 6b. Consider the system with control $u = (-1, -1 + v)$, where $v \in [-1, 3]$. We have

$$\dot Q(t) = Q(t)\left(A + (-1 + v(t))B\right) = Q(t)\left(A - B + v(t)B\right).$$

This is a control affine system with drift $X_0(Q) = Q(A-B)$ and control vector field $X_B(Q) = QB$. Let us observe that $e^{tX_0}(Q) = Qe^{t(A-B)}$. The latter can be explicitly computed, indeed we have $(A-B)^2 = -\operatorname{Id}$ and $(A-B)^3 = -(A-B)$, hence

$$\begin{aligned} e^{t(A-B)} &= \sum_{k=0}^{\infty} \frac{t^k(A-B)^k}{k!} \\ &= \left(\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!}\right)\operatorname{Id} + \left(\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!}\right)(A-B) \\ &= \cos t \operatorname{Id} + \sin t(A-B) \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}. \end{aligned}$$

This proves that X_{A-B} is a periodic vector field, and thus recurrent.

Since 0 belongs to the interior of the control set $[-1, 3]$, to complete the proof of controllability, it suffices to show that the family $\{X_0, X_B\}$ is Lie bracket generating. By question 3, we have

$$[X_0, X_B](Q) = Q[(A-B), B] = Q[A, B] = QC.$$

Since $A - B$, B and C are linearly independent, we conclude that the system is Lie bracket generating, and hence controllable. Since we used the recurrent drift theorem, we cannot conclude on the small time local controllability of the system.

Question 6c. Consider the original system with $u \in [0,1] \times [-2,2]$. Reparametrising $u = (1, -1+v)$ with $v \in [-1,3]$, we see that every trajectory of the system from Question 6b is also a trajectory of the original system (since $(1,-1+v) \in [0,1] \times [-2,2]$ for $v \in [-1,3]$). Hence the reachable set of the system from 6b is contained in the reachable set of the original system. Since the former equals $SL(2)$, we conclude that the original system is controllable.

> **Exercise 2: Time-optimal control of a particle with friction**
>
> Consider a point particle of unit mass moving on the real line, subject to linear friction and a bounded external force. Its state is described by the position $x \in \mathbb{R}$ and the velocity $v \in \mathbb{R}$. The dynamics are:
> $$\dot{x}(t) = v(t), \qquad \dot{v}(t) = -v(t) + u(t), \qquad u(t) \in [-1,1].$$
>
> 1. Write the system in the standard form $\dot{q} = f_0(q) + u\, f_1(q)$ with $q = (x,v)^\top$. Identify the drift f_0 and the controlled vector field f_1.
> 2. Using the Kalman rank condition, show that the system is controllable.
> 3. We wish to steer the system from a given initial state (x_0, v_0) to the origin $(0,0)$ in minimum time. Prove the existence of a minimal time trajectory.
> 4. Write the pre-Hamiltonian of the Pontryagin Maximum Principle for this problem, and determine the structure of the regular extremal controls.
> 5. Write the Hamiltonian equations and solve them explicitly. Show that extremal trajectories are *bang-bang* with at most one switching.
>   *Hint:* Express v as a function of x.
> 6. Sketch the optimal synthesis to the origin. What is the switching curve?

Solution **Question 1.** With $q = (x,v)^\top$, we write

$$\dot{q} = \underbrace{\begin{pmatrix} v \\ -v \end{pmatrix}}_{f_0(q)} + u \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{f_1(q)}.$$

Thus $f_0(q) = (v, -v)^\top$ is the drift and $f_1(q) = (0,1)^\top$ is the controlled vector field.

Question 2. The system is linear, of the form $\dot{q} = Aq + Bu$ with

$$A = \left.\frac{\partial f_0}{\partial q}\right|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \qquad B = f_1(0,0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We compute the Kalman matrix:

$$\mathcal{K} = [B \mid AB] = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

Since $\det \mathcal{K} = -1 \neq 0$, the Kalman rank condition is satisfied, so the system is controllable from any initial state in any time $T > 0$.

Question 3. We apply the Filippov existence theorem. The set of admissible velocities at q is

$$F(q) = \{f_0(q) + u f_1(q) : u \in [-1,1]\} = \left\{ \begin{pmatrix} v \\ -v+u \end{pmatrix} : u \in [-1,1] \right\}.$$

This set is:

- nonempty and compact (as the image of a compact set by a continuous map);
- convex (since it is an affine image of the segment $[-1,1]$).

Moreover, the target $\{(0,0)\}$ is a closed set. Finally, we need to verify that trajectories cannot escape to infinity in finite time. This follows since $f_0(q) + uf_1(q)$ is sub linear, namely there exists $C > 0$ such that $|f_0(q) + uf_1(q)| \leq C(1 + |q|)$ for all q and u. Indeed, $|f_0(q)| = \sqrt{2}|v| \leq \sqrt{2}|q|$ and $|uf_1(q)| = |u| \leq 1$.

Question 4. The pre-Hamiltonian of the time-optimal problem reads:

$$\mathcal{H}(x, v, p_x, p_v, p^0, u) = p_x v + p_v(-v + u) + p^0,$$

where $(p_x, p_v) \in \mathbb{R}^2$ is the adjoint state and $p^0 \leq 0$.

The maximisation condition yields that an extremal control u must satisfy:

$$u(t) \in \arg \max_{u \in [-1,1]} \mathcal{H}(x(t), v(t), p_x(t), p_v(t), p^0, u) = \arg \max_{u \in [-1,1]} p_v(t)u.$$

That is,

$$u(t) = \operatorname{sign}(p_v(t)) \qquad \text{whenever } p_v(t) \neq 0.$$

Question 5. The adjoint equations are:

$$\dot{p}_x = -\frac{\partial \mathcal{H}}{\partial x} = 0, \qquad \dot{p}_v = -\frac{\partial \mathcal{H}}{\partial v} = -(p_x - p_v) = -p_x + p_v.$$

From the first equation, $p_x(t) = p_x^0$ is constant. The second is a linear ODE $\dot{p}_v = p_v - p_x^0$, whose solution is:

$$p_v(t) = \left(p_v^0 - p_x^0\right) e^t + p_x^0.$$

It follows that $p_v(t)$ is monotone if $p_v^0 \neq p_x^0$, and constant if $p_v^0 = p_x^0$. In particular, wither $p_x(t) = p_v(t) = 0$ for all t (if $p_v^0 = p_x^0$), or $p_v(t)$ crosses 0 at most once. A crossing happens if and only if $0 < p_v^0 < p_x^0$ or $p_x^0 < p_v^0 < 0$.

To prove that extremal trajectories are bang-bang with at most one switching, we need to exclude the case $p_v(t) \equiv 0$. This follows since in this case $p_x(t) \equiv 0$. Thus it follows that $p^0 \neq 0$. However, the free final time condition implies that

$$0 = \mathcal{H}(x(t), v(t), p_x(t), p_v(t), p^0, u(t)) = p^0 \neq 0.$$

Hence, $p_v(t)$ cannot be identically zero, and thus extremal controls are bang-bang with at most one switching.

Question 6. For $u \equiv +1$, the system becomes

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -v + 1. \end{cases}$$

The equilibrium velocity is $v = 1$, and the phase curve can be computed by separation of variables:

$$\frac{dv}{dx} = \frac{-v+1}{v} \implies dx = \frac{v}{-v+1}\, dv = \left(-1 + \frac{1}{1-v}\right) dv, \qquad v \neq 1.$$

That is, any extremal trajectory with $u \equiv +1$ satisfies

$$x = -v - \log|1 - v| + c_+ = -v + \ln \frac{1}{|v-1|} + c_+.$$

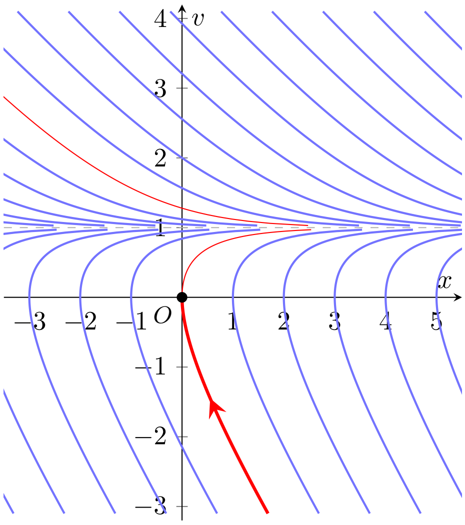
Similarly, for $u \equiv -1$, the equilibrium velocity is $v = -1$, and any extremal trajectory with $u \equiv -1$ satisfies

$$x = -v + \log|v + 1| + c_- = -v + \ln \frac{1}{|v+1|} + c_-.$$

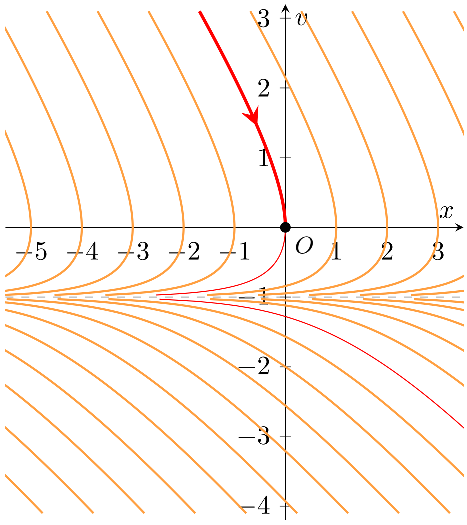
The switching curve Γ is the union of the two phase curves arriving at the origin with a single constant control:

- *Arc γ_+:* the trajectory with $u \equiv +1$ reaching $(0,0)$. Since the origin has $v = 0$ and $u = +1$ means the system was decelerating from negative velocity, this arc has $v \leq 0$ and satisfies $x = -v + \ln \frac{1}{|v-1|}$ with c_+ chosen so that $(x, v) = (0, 0)$ lies on the curve: $0 = 0 + \ln 1 + c_+$, so $c_+ = 0$. Thus:

$$\gamma_+ : \quad x = -v - \ln|v - 1|, \qquad v \leq 0.$$



(a) Phase curves for $u \equiv +1$; red arc is γ_+.



(b) Phase curves for $u \equiv -1$; red arc is γ_-.

Figure 1: Extremal trajectories in the phase plane (x, v). The red arcs $\gamma_\pm$ reach the origin and form the switching curve Γ. Arrows indicate the direction of motion.

- *Arc γ_-:* the trajectory with $u \equiv -1$ reaching $(0, 0)$. Similarly, with $v \geq 0$: $0 = 0 - \ln 1 + c_-$, so $c_- = 0$. Thus:

$$\gamma_- : \quad x = -v - \ln|v + 1|, \qquad v \geq 0.$$

The *optimal synthesis* is as follows:

- If the initial point (x_0, v_0) lies on γ_+ (with $v_0 \leq 0$) or on γ_- (with $v_0 \geq 0$), then the optimal trajectory follows this arc directly to the origin with a single constant control.
- If (x_0, v_0) lies above Γ (i.e., to the "upper-left" of the switching curve), we first apply $u = -1$ until reaching Γ (specifically γ_-), then switch to $u = +1$ and follow γ_+ to the origin.
- If (x_0, v_0) lies below Γ, we first apply $u = +1$ until reaching γ_+, then switch to $u = -1$ and follow γ_- to the origin.

In all cases, there is at most one switch, in agreement with the PMP analysis.

Exercise 3: A sub-Riemannian control system: the Grushin plane

We consider the following control system on $\mathbb{R}^2$:

$$\dot{x} = u_1, \qquad \dot{y} = x u_2, \qquad u = (u_1, u_2) \in \mathbb{R}^2.$$

For this system, we aim to solve the sub-Riemannian optimal control problem of steering from $q_{\text{in}} = (0, 0)$ to $q_{\text{fi}} \in \mathbb{R}^2$, with $u \in L^\infty([0, T]; \mathbb{R}^2)$, while minimising the cost

$$E(u) = \int_0^T (u_1^2 + u_2^2)\, dt \to \min.$$

1. Show that the system is controllable and discuss the existence of minimisers to the optimal control problem.
2. Write the Hamiltonian of the PMP and determine the structure of the regular extremal controls.
3. Discuss singular and abnormal extremals.
4. Write the Hamiltonian equations and solve them explicitly. Describe extremal trajectories. *Hint:* Justify that the initial covector satisfies $(p_x^0, p_y^0) = (\pm 1, a)$ for $a \in \mathbb{R}$.

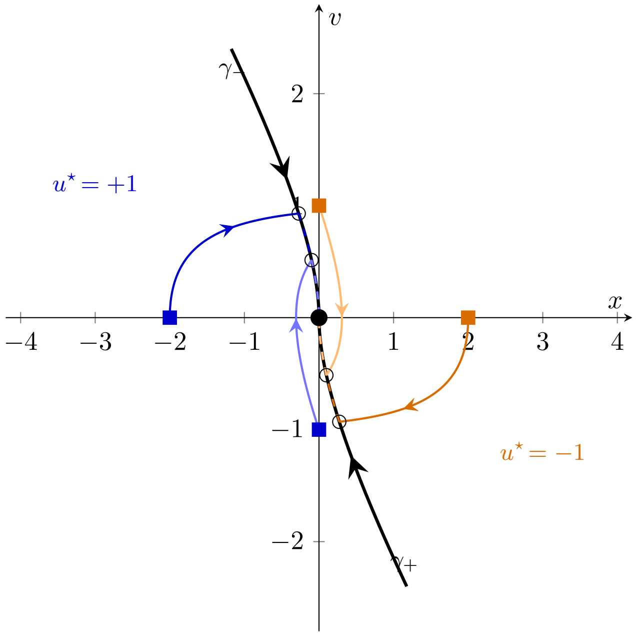


Figure 2: Optimal synthesis in the phase plane (x, v). The switching curve $\Gamma = \gamma_+ \cup \gamma_-$ (thick black, with arrows toward the origin) separates the two synthesis regions. *Blue* (■ initial condition): the optimal control is $u = +1$ (solid arc) until reaching γ_-, then $u = -1$ (dashed arc along γ_-) to the origin. *Orange* (■): the optimal control is $u = -1$ (solid) until reaching γ_+, then $u = +1$ (dashed) to the origin. Open circles mark the switching points.

> 5. Show that the trajectory corresponding to $a = 0$ is optimal for all times. Then show that for an initial covector with $a \neq 0$ there exists a time $T_a > 0$ and another covector such whose trajectory arrives at the same final point at time T_a.

Solution **Question 1.** The system writes

$$\dot{q} = u_1 f_1(q) + u_2 f_2(q), \qquad \text{where} \qquad f_1(q) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad f_2(q) = \begin{pmatrix} 0 \\ x \end{pmatrix}.$$

We have

$$[f_1, f_2](q) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence, the system is Lie bracket generating. Being symmetric (since $u \in \mathbb{R}^2$) its controllability follows by Chow-Rashevskii's theorem.

Concerning existence of minimisers, we saw during the course that this problem is equivalent to the minimal time problem with constraints $|u| \leq 1$. The set of admissible velocities at q is

$$F(q) = \{u_1 f_1(q) + u_2 f_2(q) : u \in \mathbb{R}^2\} = \{(u_1, x u_2) : u \in \mathbb{R}^2\}.$$

This set is nonempty, compact and convex for all q. Moreover, the target $\{q_{\text{fi}}\}$ is closed. Finally, trajectories cannot escape to infinity in finite time since $|u_1 f_1(q) + u_2 f_2(q)| \leq \|u\|_\infty (1 + |q|)$ is sublinear. Hence, by Filippov's existence theorem, there exists a minimiser.

Question 2. The Hamiltonian of the PMP reads:

$$\mathcal{H}(q, p, p^0, u) = \langle p, u_1 f_1(q) + u_2 f_2(q) \rangle + p^0(u_1^2 + u_2^2) = u_1 p_x + u_2 p_y x + p^0(u_1^2 + u_2^2).$$

If $p^0 \neq 0$ this is a concave functions of u, and the maximisation condition yields that an extremal control u must satisfy:

$$0 = \frac{\partial \mathcal{H}}{\partial u_1} = p_x + 2p^0 u_1, \qquad 0 = \frac{\partial \mathcal{H}}{\partial u_2} = p_y x + 2p^0 u_2.$$

Hence, regular extremal controls coincide with normal extremal controls (i.e., those with $p^0 \neq 0$). Normalizing $p^0 = -1/2$, these are given by

$$u_1 = -\frac{p_x}{2p^0} = p_x, \qquad u_2 = -\frac{p_y x}{2p^0} = p_y x.$$

Question 3. We aready established that singular extremals and abnormal extremals coincide. In this case we have $p^0 = 0$, and the maximisation condition yields $p_x = 0$ and $p_y x = 0$. Hence, singular extremals are those with $p_x = 0$ and either $p_y = 0$ or $x = 0$. Since p_y cannot vanish (otherwise $(p_x, p_y, p^0) = (0, 0, 0)$ would contradict the non-triviality condition), it must hold $x(t) = 0$ for all t. This implies that $\dot{y}(t) = 0$ for all t.

It follows that the onyl abnormal trajectory is the trivial one $(x(t), y(t)) \equiv (0, 0)$, which is not optimal for any target $q_{\text{fi}} \neq (0, 0)$.

Question 4. The maximized Hamiltonian for normal extremals is

$$\mathcal{H}(q, p) = \frac{1}{2}\left(p_x^2 + p_y^2 x^2\right).$$

It is constant along extremals, so we can normalise it to $1/2$. Hence, recallin that the initial condition is $q(0) = (0, 0)$, we have

$$\mathcal{H}(q(0), p(0), p^0, u(0)) = \frac{1}{2}(p_x^0)^2,$$

which implies that $p_x^0 = \pm 1$. On the other hand, p_y^0 can be any real number. Hence, we can parametrize the initial covector as $(p_x^0, p_y^0) = (\pm 1, a)$ for $a \in \mathbb{R}$.

In class we saw that for sub-Riemannian problems the Hamiltonian equations can be computed w.r.t. the maximized Hamiltonian $\mathcal{H}(q, p)$. The Hamiltonian equations are:

$$\begin{cases} \dot{x} = p_x, \\ \dot{y} = x^2 p_y^0, \\ \dot{p}_x = -\frac{\partial H}{\partial x} = -p_y^2 x, \\ \dot{p}_y = -\frac{\partial H}{\partial y} = 0. \end{cases}$$

Thus, $p_y(t) = p_y^0 = a$ is constant. Assume $a \neq 0$, and solve for $(x(t), p_x(t))$ finding

$$\begin{cases} x(t) = \dfrac{p_x^0}{a}\sin(at) \\ p_x(t) = p_x^0 \cos(at) \end{cases}$$

Finally, we compute y by integrating $\dot{y} = x^2 p_y^0$:

$$y(t) = \int_0^t x(s)^2 a\, ds = \frac{(p_x^0)^2}{a}\int_0^t \sin^2(as)\, ds = \frac{(p_x^0)^2}{2a}\left(t - \frac{\sin(2at)}{2a}\right).$$

If $a = 0$, we have $\dot{p}_x = 0$, so $p_x(t) = p_x^0 = \pm 1$ is constant, and thus $x(t) = \pm t$. Moreover, $\dot{y} = x^2 a = 0$, so $y(t) = 0$ for all t.

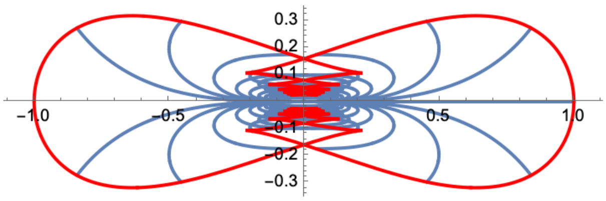


Figure 3: Extremal trajectories in the plane (x, y). The red arc reprensents the final position of extremal trajectories at time $T = 1$ as a function of the parameter a. The blue arcs are extremal trajectories for different values of a.

Question 5. The final position at time T of an extremal trajectory with initial covector $(p_x^0, p_y^0) = (\pm 1, a)$ is given by

$$(x(T), y(T)) = \begin{cases} \left(\pm\frac{1}{a}\sin(aT), \frac{1}{2a}\left(T - \frac{\sin(2aT)}{2a}\right)\right), & a \neq 0 \\ (\pm T, 0), & a = 0. \end{cases}$$

Observe that $\dot{y} = x^2 a$ has fixed sign, so no extremal trajectory with $a \neq 0$ can cross the x-axis. In particular, the trajectory with $a = 0$ is optimal for all times.

Concerning $a \neq 0$, let $(x_\pm(t), y_\pm(t))$ be the extremal trajectory with initial covector $(p_x^0, p_y^0) = (\pm 1, a)$. We have that $y_+(t) = y_-(t)$ for all t, and $x_+(t) = -x_-(t)$ for all t. Hence, the two trajectories are symmetric with respect to the y-axis. Moreover, they intersect at time T if and only if $\sin(aT) = 0$, i.e., if and only if $aT = k\pi$ for some integer k. At this time, the two trajectories reach the same point on the y-axis, which is given by

$$(x_\pm(T), y_\pm(T)) = (0, \frac{T}{2a}).$$

We conclude that if $a \neq 0$ the trajectory with initial covector $(p_x^0, p_y^0) = (1, a)$ and the trajectory with initial covector $(p_x^0, p_y^0) = (-1, a)$ reach the same point at time $T_a = \frac{\pi}{|a|}$.