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Introduction to spectral geometry

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Preface

These are the lecture notes for a course given at the CIMPA summer school "Partial Differential Equations (PDEs) and Calculus of Variations" at Dangbo (Benin), from 29/08/2022 to 09/09/2022.

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Chapter 1

Introduction

Spectral geometry is the study of the interconnections between the geometry of a “space” (e.g., a Riemannian manifold) and the spectral properties of a “natural” operator acting on this space (e.g., the Laplace-Beltrami operator on the manifold).

By *geometry*, we mean properties like the volume, the diameter, the curvature, etc. On the other hand, the *spectral properties* we are tipically most interested concern the distribution of eigenvalues.

In this lectures we will consider two types of problems:

Direct problems knowing the “shape” of the space, can we say something on its “sound”, i.e., his *characteristic frequencies*?

Inverse problems Can we say something about the “shape” of the space if we know its “sound”? In particular, if two domains have the same spectrum (they are *isospectral*) are they identical (in a sense to be precised)?

In the following we will mainly consider the (Dirichlet) Laplace operator $\varDelta$ on a bounded domain $\varOmega \subset \mathbb{R}^d$. This is the operator given by

$$\varDelta = -\sum_{i=1}^{d} \partial_{x_i}^2, \qquad \mathrm{Dom}(\varDelta) = H^2(\varOmega) \cap H_0^1(\varOmega). \tag{1.1}$$

A note of attention: the above definition contains a minus sign, which guarantees the positiveness of the operator (i.e., $(\varDelta u, u) \geq 0$ for all $u \in \mathrm{Dom}(\varDelta)$). This convention is opposite to the one usually encountered in analysis, but is fairly standard in geometry.

Chapter 2

Functional analytic preliminaries

We start by recalling some preliminaries in Sobolev spaces and operator theory. We refer to [7, 1] for details and proof.

We denote the Euclidean scalar product and the Euclidean norm on $\mathbb{R}^d$ by $\langle\cdot,\cdot\rangle_{\mathbb{R}^d}$ and $|\cdot|_{\mathbb{R}^d}$, respectively.

Functional spaces

Let $\Omega\subset\mathbb{R}^d$ be an open set, we denote by $L^2(\Omega)$ the space of square integrable functions $u:\Omega\to\mathbb{R}$. This is an Hilbert space when endowed with the scalar product and associated norm:

$$\langle u,v\rangle_{L^2(\Omega)}=\int_\Omega u\,v \qquad \text{and} \qquad \|u\|_{L^2(\Omega)}=\int_\Omega |u|^2, \qquad u,v\in L^2(\Omega).$$

Given a locally integrable function $u:\Omega\to\mathbb{R}$, its gradient ∇u is defined in the sense of distributions as the $\mathbb{R}^d$-valued distribution satisfying

$$\langle\nabla u,X\rangle=-\int_\Omega u\operatorname{div}X, \qquad \text{for any } X\in C_c^\infty(\Omega;\mathbb{R}^d).$$

Here, we are using $\langle\cdot,\cdot\rangle$ to denote (with a slight abuse of notation) the duality product on $C_c^\infty(\Omega;\mathbb{R}^d)$. That is, if $\nabla u\in C^\infty(\Omega;\mathbb{R}^d)$, we have

$$\langle\nabla u,X\rangle=\int_\Omega\langle\nabla u(x),X(x)\rangle_{\mathbb{R}^d}\,dx, \qquad \text{for any } X\in C_c^\infty(\Omega;\mathbb{R}^d).$$

Similarly, we can define the distributional laplacian Δu (see the expression (1.1)) by

$$\langle\Delta u,\varphi\rangle=-\int_\Omega u\,\Delta X, \qquad \text{for any } \varphi\in C_c^\infty(\Omega).$$

The Sobolev space $H^1(\Omega)$ is then defined as

$$H^1(\Omega)=\left\{u\in L^2(\Omega)\mid|\nabla u|_{\mathbb{R}^d}\in L^2(\Omega)\right\}. \tag{2.1}$$

This is an Hilbert space, when endowed with the following scalar product and associated norm:

$$\langle u,v\rangle_{H^1(\Omega)}=\langle u,v\rangle_{L^2(\Omega)}+\int_\Omega\langle\nabla u,\nabla v\rangle_{\mathbb{R}^d},$$
$$\|u\|_{H^1(\Omega)}=\|u\|_{L^2(\Omega)}+\|\nabla u\|_{L^2(\Omega)}.$$

Clearly $H^1(\Omega) \subset L^2(\Omega)$, and it is possible to show that $C^\infty(\Omega) \cap H^1(\Omega)$ is dense in $H^1(\Omega)$.

We will also need the space $H^1_0(\Omega)$, that is the closure of $C_c^\infty(\Omega)$ w.r.t. to the norm (2.1). We recall that $H^1_0(\mathbb{R}^d) = H^1(\mathbb{R}^d)$, but this is false in general.

Finally, we introduce the space $H^2(\Omega)$ which is the space of square integrable functions in $L^2(\Omega)$ such that both $|\nabla u|_{\mathbb{R}^d}$ and all second partial derivatives $\partial_{x_i}\partial_{x_j} u$ are square integrable. This is also an Hilbert space, with norm

$$\|u\|_{H^2(\Omega)} = \|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} + \|\operatorname{Hess} u\|_{L^2(\Omega)}.$$

Here, $\operatorname{Hess} u \in \mathbb{R}^{d\times d}$ denotes the Hessian matrix of u (i.e., $(\operatorname{Hess} u)_{i,j} = \partial_{x_i}\partial_{x_j} u$) and we are considering the Euclidean norm on $\mathbb{R}^{d\times d}$. That is,

$$\|\operatorname{Hess} u\|_{L^2(\Omega)} = \sum_{i,j=1}^{d} \int_\Omega |\partial_{x_i}\partial_{x_j} u|^2. \tag{2.2}$$

One can show that $C^\infty(\Omega) \cap H^2(\Omega)$ is dense in $H^2(\Omega)$.

When restricting to consider $\operatorname{Dom}(\Delta) = H^1_0(\Omega) \cap H^2(\Omega)$, one can actually replace the above norm by a simpler one, as we show in the following.

Theorem 2.1. *There exists $c > 0$ such that*

$$\|u\|_{H^2(\Omega)} \le c \left(\|u\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega)} \right), \qquad u \in H^1_0(\Omega) \cap H^2(\Omega). \tag{2.3}$$

Proof. We have to show that both $\|\nabla u\|_{L^2}(\Omega)$ and $\|\operatorname{Hess} u\|_{L^2(\Omega)}$ can be controlled by the quantity on the r.h.s. of (2.3). Observe that, by density of $C_c^\infty(\Omega)$ in $H^1_0(\Omega) \cap H^2(\Omega)$, it suffices to prove this for $u \in C_c^\infty(\Omega)$.

We start by considering $\|\nabla u\|_{L^2(\Omega)}$. By Green formula, letting $\vec\nu$ be the outward pointing normal to Ω we have

$$\|\nabla u\|^2_{L^2(\Omega)} = \int_\Omega u\, \Delta u + \int_\Omega u \langle \nabla u, \vec\nu\rangle_{\mathbb{R}^d} = \int_\Omega u\, \Delta u.$$

Here, in the last passage we used the fact that $u|_{\partial\Omega} \equiv 0$, since u is compactly supported in Ω. Then, by Cauchy-Schwarz and Young's inequality[1] we have

$$\|\nabla u\|^2_{L^2(\Omega)} \le \frac{1}{2} \left(\|u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)} \right).$$

The desired bound (without the squares) follows from the inequality $\sqrt{a^2+b^2} \le |a| + |b|$.

We now turn to an argument for the bound on $\|\operatorname{Hess} u\|_{L^2(\Omega)}$. Similarly as above, integrating by parts the terms in expression (2.2), we have

$$\int_\Omega |\partial_{x_i}\partial_{x_j} u|^2 = -\int_\Omega \partial_{x_j} u\, \partial^2_{x_i} \partial_{x_j} u = \int_\Omega \partial^2_{x_j} u\, \partial^2_{x_i} u.$$

[1]That is, $ab \le (a^2 + b^2)$ if $a, b > 0$.

Summing up w.r.t. $i, j \in \{1, \dots, d\}$ we finally get

$$\| \operatorname{Hess} u\|_{L^2(\Omega)}^2 = \int_\Omega |\Delta u|^2.$$

□

The above theorem guarantees that $H_0^1(\Omega) \cap H^2(\Omega)$ is an Hilbert space, whose inherited norm from $H^2(\Omega)$ is equivalent to

$$\|u\|_{H_0^1(\Omega)\cap H^2(\Omega)} = \|u\|_{L^2(\Omega)}^2 + \|\Delta u\|_{L^2(\Omega)}^2.$$

The case of a bounded set

If Ω is a bounded set, the Sobolev spaces previously introduced have many additional properties.

The first important result is the following, see [1, Theorem 9.16].

Theorem 2.2 (Rellich-Kondrachov Theorem). *Let $\Omega \subset \mathbb{R}^d$ be a bounded set of class*[2] *C^1. Then, the injection of $H^1(\Omega)$ into $L^2(\Omega)$ is compact. That is, any bounded sequence $\{u_n\}_n \subset H^1(\Omega)$ (i.e., such that $\|u_n\|_{H^1(\Omega)} \le c$ for some $c > 0$) admits a convergent subsequence in $L^2(\Omega)$.*

Let us now look more closely at $H_0^1(\Omega)$. It turns out that, when Ω is bounded, the $L^2(\Omega)$ norm of the gradient allows to bound the $L^2(\Omega)$ norm of the function. This is the content of the following celebrated result, see [1, Corollary 9.19].

Theorem 2.3 (Poincaré's Inequality). *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. Then there exists a constant $c > 0$ (depending on Ω) such that*

$$\|u\|_{L^2(\Omega)} \le c \int_\Omega |\nabla u|_{\mathbb{R}^d}^2, \qquad \text{for any } u \in H_0^1(\Omega).$$

Remark 2.4. The above inequality is trivially false for the whole $H^1(\Omega)$, since the constant function $u(x) = 1$, $x \in \Omega$, belongs to $H^1(\Omega)$ but $\int_\Omega |\nabla u|_{\mathbb{R}^d}^2 = 0$ while[3] $\|u\|_{L^2(\Omega)} = |\Omega|$.

Poincaré inequality implies that the Hilbert structure on $H_0^1(\Omega)$ inherited from $H^1(\Omega)$, is equivalent to the one given by the scalar product and norm:

$$\langle u, v\rangle_{H_0^1(\Omega)} = \int_\Omega \langle \nabla u, \nabla v\rangle_{\mathbb{R}^d}, \qquad \text{and} \qquad \|u\|_{H_0^1(\Omega)} = \|\nabla u\|_{L^2(\Omega)}.$$

In particular, this means that, in order to apply the Rellich-Kondrachov Theorem to a sequence of functions in $H_0^1(\Omega)$, it suffices to check that $\|\nabla u_n\|_{L^2(\Omega)}$ be uniformly bounded w.r.t. n.

[2] We recall that Ω is of class C^k if for any $x \in \partial\Omega$ there exists a neighborhood $U \subset \mathbb{R}^d$ of x and a boundary defining submersion $F : U \to \mathbb{R}$ of class C^k. That is, $\nabla F \neq 0$ on U, and $\partial\Omega \cap U = F^{-1}(0)$.

[3] Here, we are denoting by $|\Omega|$ the Lesbegue measure of Ω.

Laplace operator

Let us briefly discuss the definition of the Laplace operator (and in particular its domain) given in the Introduction. For the following discussion, we refer mainly to [7].

The Laplace operator (1.1) appears in a variety of contexts, and of particular interest is the Poisson equation

$$\varDelta u = f \qquad \text{on } \varOmega. \tag{2.4}$$

This equation arises, for example, in electrostatics, where u is the electric potential induced by a given charge distribution f.

If $f \in L^2(\varOmega)$, in order to give a formal setting to the above equations it is natural to consider $\varDelta$ as an operator on $L^2(\varOmega)$. Solving equation (2.4) amounts to invert this operator.

Being a differential operator it is not difficult to check that $\varDelta$ is unbounded and hence it cannot be defined on the whole space $L^2(\varOmega)$. The typical approach is to consider the operator $\varDelta$ given by expression (2.4), with domain $\mathrm{Dom}(\varDelta) = C_c^\infty(\varOmega)$. The resulting operator is a densely-defined unbounded linear operator on $L^2(\varOmega)$. By Green formula and Poincaré Inequality, $\varDelta$ is also *symmetric* and *strictly positive*, meaning that

$$\begin{aligned} &\langle \varDelta u, v\rangle_{L^2(\varOmega)} = \langle u, \varDelta v\rangle_{L^2(\varOmega)}, \qquad \text{for any } u, v \in C_c^\infty(\varOmega), \\ &\exists c > 0 :\ \langle \varDelta u, u\rangle \geq c\|u\|_{L^2(\varOmega)}, \qquad \text{for any } u \in C_c^\infty(\varOmega). \end{aligned} \tag{2.5}$$

However, this operator is not *self-adjont*, i.e., it does not coincide with its adjoint.

Recall that the adjoint $\varDelta^*$, roughly speaking, is the operator with the "largest domain" such that[4]

$$\langle \varDelta u, v\rangle_{L^2(\varOmega)} = \langle u, \varDelta^* v\rangle_{L^2(\varOmega)}, \qquad \text{for any } u \in \mathrm{Dom}(\varDelta),\ v \in \mathrm{Dom}(\varDelta^*).$$

Clearly $\mathrm{Dom}(\varDelta^*) \supset \mathrm{Dom}(\varDelta)$ by (2.5), and their definition coincides there. However, $\varDelta \neq \varDelta^*$. Indeed, by Green formula it is fairly easy to verify that

$$\mathrm{Dom}(\varDelta^*) = H^2(\varOmega) \supsetneq C_c^\infty(\varOmega) = \mathrm{Dom}(\varDelta).$$

The lack of self-adjointness is problematic, due to the following result. See [7, Theorem VIII.3] for a general version. Here, we present a version adapted to the strict positivity of $\varDelta$, which can be derived using also the Corollary after [7, Theorem X.1].

Theorem 2.5 (Basic criterion of self adjointness)**.** *Let T be a densely-defined symmetric operator on an Hilbert space H. Assume that there exists $c > 0$ such that*

$$\langle Tu, u\rangle_H \geq c\|u\|_H, \qquad \textit{for all } u \in D(T).$$

Then, the following are equivalent:

[4] Actually, one defines $\mathrm{Dom}(\varDelta^*)$ as the set of $v \in L^2(\varOmega)$ such that there exists $z \in L^2(\varOmega)$ for which $\langle \varDelta u, v\rangle_{L^2(\varOmega)} = \langle u, z\rangle_{L^2(\varOmega)}$, for any $u \in \mathrm{Dom}(\varDelta)$.

a. *T is self-adjoint;*

b. *T is closed*[5] *and* $\ker T^* = \{0\}$*;*

c. $\operatorname{range} T = H$.

One should read this result as an existence and uniquess result for the Poisson equation for the operator T:

$$Tu = f, \qquad f \in H. \tag{2.6}$$

Indeed, assume that T is self-adjoint (i.e., *a.* holds). Then, *b.* yields the uniqueness of solutions to (2.6), since

$$Tu = f = Tv \implies T(u - v) = 0 \implies u - v \in \ker T.$$

On the other hand, *c.* yields the existence of solutions, since it is exactly saying that for any $f \in H$ there exists $u \in \operatorname{Dom} T$ such that $Tu = f$.

Since Δ with domain $C_c^\infty(\Omega)$ is not self-adjoint, we thus have that for this operator there is no existence nor uniqueness of solutions to (2.4)!

It turns out that it is possible to *extend* the domain of definition of Δ (which *reduces* the domain of its adjoint) in order to obtain a self-adjoint operator. For the following, we are only interested in two particular extensions of Δ with domain $C_c^\infty(\Omega)$, the Dirichlet Laplace operator Δ_D (that we will still denote by Δ when no confusion arises) and the Neumann Laplace operator Δ_N. Their respective domains are,

$$\operatorname{Dom}(\Delta_D) = H_0^1(\Omega) \cap H^2(\Omega),$$
$$\operatorname{Dom}(\Delta_N) = \left\{ u \in H^2(\Omega) \mid \frac{\partial u}{\partial \vec{\nu}} = 0 \text{ on } \partial\Omega \right\}.$$

Remark 2.6. To define the domain of Δ_N one should prove a trace theorem guaranteeing that the normal derivative at the boundary is well-defined. This can be cumbersome, so the usual approach is to define Δ_N via the form formalism [7, Section XIII.15].

Solving Poisson equation (2.4) with $f \in L^2(\Omega)$ for the operators Δ_D or Δ_N, amounts to solving the classical Cauchy problems with Dirichlet or Neumann boundary conditions, respectively,

$$\text{(D)} \begin{cases} \Delta u = f \text{ on } \Omega \\ u|_{\partial\Omega} \equiv 0 \end{cases} \qquad \text{(N)} \begin{cases} \Delta u = f \text{ on } \Omega \\ \frac{\partial u}{\partial \vec{\nu}}|_{\partial\Omega} \equiv 0 \end{cases}$$

Exercices

2.1 Show that Δ with domain $C_c^\infty((0,1))$ is an unbounded operator on $L^2((0,1))$

[5] That is, its graph $\{(u, Tu) \mid u \in H\} \subset H \times H$ is a closed subset of $H \times H$.

Chapter 3

Spectral theory of the Laplace operator

Let $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) be an bounded open set with sufficiently regular boundary (say, C^2). Consider the Laplace operator Δ defined in (1.1), which is an unbounded operator on the state space $L^2(\Omega)$.

We stress that, due to its domain, this operator is considered with Dirichlet boundary conditions. Since later on we will need to consider also Neumann boundary conditions, when needed we will denote the Laplace operator with Dirichlet boundary conditions as Δ_D and the one with Neumann boundary conditions as Δ_N

Definition and existence of the spectrum

In order to discuss the spectrum of Δ, we must first show that it exists, and that it is composed of discrete eigenvalues (i.e., it is a purely discrete spectrum).

Theorem 3.1. *There exists a sequence of positive real numbers accumulating at infinity*

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \longrightarrow +\infty,$$

and a sequence of functions $\{\varphi_n\}_{n=1}^{+\infty} \subset L^2(\Omega)$ such that:

- *The family $\{\varphi_n\}_{n=1}^{+\infty}$ is an Hilbert basis of $L^2(\Omega)$ and, moreover, $\varphi_n \in C^\infty(\Omega) \cap \mathrm{Dom}(\Delta)$;*
- *each λ_n is an eigenvalue for Δ associated with the eigenfunction φ_n:*

$$\Delta\varphi_n = \lambda_n\varphi_n, \qquad \forall n \in \mathbb{N}^*.$$

The positive real numbers λ_n introduced in the previous theorem, are the eigenvalues of Δ on Ω, while the φ_n are the associated eigenfunctions. The set $\{\lambda_n\}_{n=1}^{+\infty}$ is the *spectrum of Δ on Ω* and is denoted $\mathrm{spec}(\Delta)$.

The proof of Theorem 3.1 consists of the following steps:

a. Show that the inverse Δ^{-1} of the Laplacian is a well-defined compact operator (in particular, it is bounded and $\mathrm{Dom}(\Delta^{-1}) = L^2(\Omega)$).

b. Apply the spectral theorem for compact operators, which guarantees that $\mathrm{spec}(\Delta^{-1})$ is discrete.

c. Show that the elements of the spectrum of Δ are the inverses of the elements in $\mathrm{spec}(\Delta^{-1})$.

In order to perform the previous steps, we need to admit two classical results in functional analysis (see [1, Cor. 5.8] and [7, VII]).

Theorem 3.2 (Lax-Milgram)**.** *Let $(H, \langle\cdot,\cdot\rangle)$ a real Hilbert space, and let $a : H \times H \to \mathbb{R}$ be a bilinear form on H that is*

- *continuous, i.e., there exists $c > 0$ such that $a(u,v) \le c\|u\|_H\|v\|_H$ for all $u, v \in H$;*
- *coercive, i.e., there exists $\alpha > 0$ such that $a(u,u) \ge \alpha\|u\|_H$ for all $u \in H$.*

Then, for any continuous linear form $\ell \in H'$ on H there exists a unique $u \in H$ such that

$$\ell(v) = a(u,v), \qquad \forall v \in H.$$

Theorem 3.3 (Spectral theorem)**.** *Let $(H, \langle\cdot,\cdot\rangle)$ be an infinite dimensional real Hilbert space, and let $B : H \to H$ be a bounded linear operator on H. Assume moreover that:*

- *B is self-adjoint, i.e., $\langle Bu, u\rangle = \langle u, Bu\rangle$ for all $u, v \in H$;*
- *B is compact, i.e., the image of the unit ball of H under B is relatively compact;*
- *B is positive definite, i.e., $\langle Bu, u\rangle > 0$ for all $u \in H \setminus \{0\}$.*

Then, there exist a sequence $(\mu_n)_{n=1}^{+\infty} \subset \mathbb{R}$ and a sequence $(\psi_n)_{n=1}^{+\infty} \subset H$ such that

$$B\psi_n = \mu_n\psi_n, \qquad \mu_n > 0, \qquad \mu_n \to 0 \text{ as } n \to +\infty.$$

Proof of Theorem 3.1. Recall the Green formula,

$$\langle \Delta u, v\rangle_{L^2(\Omega)} = \int_\Omega \nabla u \cdot \nabla v, \qquad \text{for all } u \in \mathrm{Dom}(\Delta), v \in H_0^1(\Omega). \tag{3.1}$$

Constructing Δ^{-1}. Let $f \in L^2(\Omega)$. Observe that in order to define $\Delta^{-1}f$, we have to find $u \in \mathrm{Dom}(\Delta)$ such that $\Delta u = f$. Let us define the following bilinear form

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \qquad u, v \in H_0^1(\Omega).$$

The essential observation is that, by Green formula (3.1), $u \in H_0^1(\Omega)$ is such that $\Delta u = f$ (i.e., $u = \Delta^{-1}f$) if and only if

$$\langle f, v\rangle_{L^2(\Omega)} = a(u,v), \qquad \text{for any } v \in H_0^1(\Omega).$$

Since $\ell(v) := \langle f, v\rangle_{L^2(\Omega)}$ is a continuous linear form[1], we will apply the Lax-Milgram Theorem in order to guarantee the existence of such u.

[1]Continuity follows from Cauchy-Schwarz: $\ell(v) \le \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}$ for any $v \in L^2(\Omega)$.

Since[2] $H^1_0(\Omega)$ is an Hilbert space with scalar product $\langle u, v\rangle = a(u, v)$, we will apply the Lax-Milgram Theorem with $H = H^1_0(\Omega)$. Let us verify the assumptions on a:

- The fact that a is a bilinear form on $H^1_0(\Omega)$ is evident (it is indeed also symmetric).
- The continuity is a consequence of Cauchy-Schwarz. Indeed, let $u, v \in H^1_0(\Omega)$, then

$$|a(u, v)| = \|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} = \|u\|_{H^1(\Omega)}\|v\|_{H^1_0(\Omega)}.$$

- The coercivity is immediate since $a(u, u) = \|\nabla u\|^2_{L^2(\Omega)} = \|u\|_{H^1_0(\Omega)}$.

Since a satisfies the assumptions of the Lax-Milgram Theorem, we have that there exists $u \in H^1_0(\Omega)$ such that $\ell(\cdot) = a(u, \cdot)$ as forms on $H^1_0(\Omega)$. However, observe that $C^\infty_c(\Omega) \subset H^1_0(\Omega)$, so evaluating the previous equality on any $v \in C^\infty_c(\Omega)$ we can perform an additional integration by parts to obtain

$$\int_\Omega f\, v = \int_\Omega u\, \Delta v, \qquad \text{for any } v \in C^\infty_c(\Omega).$$

This amounts to the distributional definition of $f = \Delta u$. In particular, $\Delta u = f \in L^2(\Omega)$ and thus $u \in H^2(\Omega)$, which yields $u \in \mathrm{Dom}(\Delta)$. (Here, we are using the C^2 regularity of the boundary of Ω and classical elliptic regularity results.)

This finally allows us to define $\Delta^{-1} : L^2(\Omega) \to L^2(\Omega)$ as the operator that associates to each $f \in L^2(\Omega)$ the function $u \in \mathrm{Dom}(\Delta) \subset L^2(\Omega)$ thus obtained.

Properties of Δ^{-1}. We now check that Δ^{-1} satisfies all the assumptions of Theorem 3.3. Clearly, Δ^{-1} is linear, being the inverse of a linear operator. Then,

- Δ^{-1} is positive definite: Letting $f = \Delta u$, this follows from

$$\langle \Delta^{-1} f, f\rangle_{L^2(\Omega)} = \langle u, f\rangle_{L^2(\Omega)} = \ell(u) = a(u, u) = \|\nabla u\|_{L^2(\Omega)}.$$

  Since $\|\nabla u\|_{L^2(\Omega)}$ is a norm on $H^1_0(\Omega)$ it vanishes only if $u \equiv 0$. Since this implies that $f = \Delta u \equiv 0$, this shows that $\langle \Delta^{-1} f, f\rangle > 0$ as soon as $f \not\equiv 0$.

- Δ^{-1} is self-adjoint: Let $f, g \in L^2(\Omega)$ and define $u = \Delta^{-1} f$ and $\tilde{u} = \Delta^{-1} g$. Then, by definition,

$$\langle f, v\rangle = a(u, v) \text{ and } \langle g, v\rangle = a(\tilde{u}, v), \qquad \forall v \in H^1_0(\Omega).$$

  In particular,

$$\langle \Delta^{-1} f, g\rangle = \langle u, g\rangle = a(\tilde{u}, u) = \langle f, \tilde{u}\rangle = \langle f, \Delta^{-1} g\rangle.$$

[2]The fact that $a(u, u) = 0$ implies $u \equiv 0$ is a consequence of the Poincaré inequality [1, Corollary 9.19]: there exists $c > 0$ such that $\|u\|_{L^2(\Omega)} \le c\|\nabla u\|_{L^2(\Omega)}$ for all $u \in H^1_0(\Omega)$.

- Δ^{-1} is compact: In order to prove that Δ^{-1} is compact, it suffices to show that the image of a bounded sequence $(f_n)_n \subset L^2(\Omega)$ admits a convergent subsequence in $L^2(\Omega)$. Without loss of generality assume that $\|f_n\|_{L^2(\Omega)} \leq 1$. Recall that by construction $\Delta^{-1} f_n \in H_0^1(\Omega)$. Since Ω is bounded, by the Rellich-Kondrachov Theorem [1, Theorem 9.16] the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact and thus it suffices to show that $(\Delta^{-1} f_n)_n$ is a bounded sequence in $H_0^1(\Omega)$. To this aim, we let $u_n = \Delta^{-1} f_n$ and compute

$$\begin{aligned}\|u_n\|^2_{H_0^1(\Omega)} = \|\nabla u_n\|^2_{L^2(\Omega)} = a(u_n, u_n)\\ = \int_\Omega f_n\, u_n \leq \|f_n\|_{L^2(\Omega)} \|u_n\|_{L^2(\Omega)}\end{aligned}$$

  Here, from the the second to the third line we used the definition of Δ^{-1} given by the Lax-Milgram Theorem, and then applied the Cauchy-Schwarz Inequality. Recalling that $\|f_n\|_{L^2(\Omega)} \leq 1$ and that $\|u_n\|_{L^2(\Omega)} \leq c\|u_n\|_{H_0^1(\Omega)}$ by the Poincaré inequality, we finally have

$$\|u_n\|^2_{H_0^1(\Omega)} \leq \|u_n\|_{H_0^1(\Omega)} \implies \|u_n\|_{H_0^1(\Omega)} \leq 1.$$

  This proves that $(\Delta^{-1} f_n)_n$ is a bounded sequence in $H_0^1(\Omega)$, thus showing that Δ^{-1} is compact.

Construction of the spectrum of Δ. We have shown that the Spectral Theorem (Theorem 3.3) can be applied to Δ^{-1}. We thus have $(\mu_n)_n \subset (0, +\infty)$ such that $\mu_n \to 0$ and an Hilbert basis $(\psi_n) \subset L^2(\Omega)$ such that

$$\Delta^{-1}\psi_n = \mu_n \psi_n \qquad \text{for all } n \in \mathbb{N}^*. \tag{3.2}$$

Let $u_n := \Delta^{-1}\psi_n$. We have shown that $u_n \in \mathrm{Dom}(\Delta)$ and, by definition of μ_n, it holds $u_n = \mu_n \psi_n$. This shows that $\psi_n \in \mathrm{Dom}(\Delta)$, and thus it suffices to apply Δ on both sides of (3.2) to show that $\lambda_n := \mu_n^{-1}$ is an eigenvalue of Δ with eigenfunction ψ_n.

To have that $\mathrm{spec}(\Delta) = \{\lambda_n\}_{n=1}^{+\infty}$ it suffices to observe that if $\Delta u = \eta u$ for some $u \in \mathrm{Dom}(\Delta)$ and $\eta \in \mathbb{R}$, applying Δ^{-1} on both sides would yield that $\eta = \mu_n$ for some $n \in \mathbb{N}^*$. Finally, to conclude the proof, we observe that standard elliptic regularity results allow to show that $\psi_n \in C^\infty(\Omega)$, and that $\lambda_n \to +\infty$ since $\mu_n \to 0$. □

Exercices

3.1 Prove this theorem.

3.2 Compute the eigenvalues of Δ on the interval $\Omega = (0, 1) \subset \mathbb{R}$.

3.3 Compute the eigenvalues of Δ on the cube $\Omega = \Pi_{i=1}^d (0, 1) \subset \mathbb{R}^d$.

Min-max theorem

Solving exactly the eigenvalues equation is typically impossible. To get informations on the distribution of the eigenvalue, one usually employs a variational technique.

Definition 3.4. The *Rayleigh quotient* associated with $u \in H_0^1(\Omega)$ is the quantity

$$R(u) = \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2}.$$

For an eigenvalue $\lambda \in \operatorname{spec}(\Delta)$, we denote by E_λ its eigenspace.

Theorem 3.5 (Courant-Fischer max-min theorem)**.** *We have the following:*

- *It holds*
$$\lambda_n = \inf_{u \in H_{n-1}} R(u) = \sup_{u \in E_{\lambda_n}} R(u), \tag{3.3}$$
where $H_{n-1} = H_0^1(\Omega) \cap E_{\lambda_n}^\perp$.
- *It holds*
$$\lambda_n = \inf_{X \in \Phi_n(H_0^1(\Omega))} \sup_{u \in X} R(u), \tag{3.4}$$
where $\Phi_k(H_0^1(\Omega)) = \{n - \textit{dimensional linear subspaces} X \subset H_0^1(\Omega)\}$.

Proof. We start by proving the first part of the statemnt. Given $u \in H_0^1(\Omega) \setminus \{0\}$, we decompose it on the Hilbert basis of the eigenfunctions $\{\varphi_n\}_{n=1}^{+\infty}$:

$$u(x) = \sum_{i=1}^{+\infty} a_n \varphi_n(x), \qquad a_i \in \mathbb{R}.$$

If $u \in \operatorname{Dom}(\Delta)$, by the Green formula (3.1) we have

$$\int_\Omega |\nabla u|^2 = \int_\Omega u\,\Delta u = \left\langle \sum_{i=1}^{+\infty} a_i \varphi_i, \sum_{i=1}^{+\infty} \lambda_i a_i \varphi_i \right\rangle_{L^2(\Omega)} = \sum_{i=1}^{+\infty} \lambda_i a_i^2.$$

By density of $\operatorname{Dom}(\Delta)$ in $H_0^1(\Omega)$, one can show that the above formula holds for all $u \in H_0^1(\Omega)$. If $u \in H_{n-1}$, then $a_k = 0$ for $k = 0, \dots, n-1$. Consequently,

$$R(u) = \frac{\sum_{i=n}^{+\infty} \lambda_i a_i^2}{\sum_{i=n}^{+\infty} a_i^2} \geq \lambda_n \frac{\sum_{i=n}^{+\infty} \lambda_i a_i^2}{\sum_{i=n}^{+\infty} a_i^2} = \lambda_n \implies \min_{u \in H_{n-1}} R(u) \geq \lambda_n.$$

To complete the proof of (3.5) it suffices to observe that $\varphi_n \in H_{n-1}$ and that $R(\varphi_n) = \lambda_n$, so that the infimum is indeed attained by a function of H_{n-1} and thus we have equality.

We now turn to an argument for (3.6). As $E_{\lambda_n} \in \Phi_n(H_0^1(\Omega))$, by (3.5) we have the inequality

$$\lambda_n = \sup_{u\in E_{\lambda_n}} R(u) \geq \inf_{X\in\Phi_n(H_0^1(\Omega))} \sup_{u\in X} R(u).$$

Conversely, for any $X \in \Phi_n(H_0^1(\Omega))$, a dimensional argument shows that there is a non-zero $v \in X \cap H_{n-1}$. Hence,

$$R(v) \geq \inf_{u\in H_{n-1}} R(u) = \lambda_n \implies \sup_{v\in X} R(v) \geq \lambda_n,$$

proving the statement. □

We will denote by $\lambda_n(\Omega)$ the eigenvalues of Δ on the domain Ω.

Corollary 3.6. *Let $\Omega_1 \subset \Omega_2$ be two bounded domains. Then, for any $n \in \mathbb{N}^*$ it holds*

$$\lambda_n(\Omega_1) \geq \lambda_n(\Omega_2).$$

Exercices

3.4 What is the connection between λ_1 and the constant in the Poincaré inequality?

3.5 Prove Corollary 3.6.

Neumann boundary conditions

Under the regularity assumptions on Ω (i.e., at least C^2) the outward pointing normal vector $\vec{\nu}$ is well-defined. The Neumann Laplace operator is Δ_N, defined by the same formula (1.1), with domain

$$\mathrm{Dom}(\Delta_N) = \left\{ u \in H^2(\Omega) \mid \frac{\partial u}{\partial \vec{\nu}} = 0 \text{ on } \partial\Omega \right\}.$$

One can replicate the proof of Theorem 3.1, changing the domain of definition of the form a from $H_0^1(\Omega)$ to $H^1(\Omega)$, to obtain the following.

Theorem 3.7. *There exists a sequence of positive real numbers accumulating at infinity*

$$0 \leq \mu_1 \leq \mu_2 \leq \ldots \longrightarrow +\infty,$$

and a sequence of functions $\{\varphi_n\}_{n=1}^{+\infty} \subset L^2(\Omega)$ such that:

- *The family $\{\varphi_n\}_{n=1}^{+\infty}$ is an Hilbert basis of $L^2(\Omega)$ and, moreover, $\varphi_n \in C^\infty(\Omega) \cap \mathrm{Dom}(\Delta)$;*
- *each μ_n is an eigenvalue for Δ_N associated with the eigenfunction φ_n:*

$$\Delta\varphi_n = \mu_n\varphi_n, \qquad \forall n \in \mathbb{N}^*.$$

Moreover, $\mu_1 = 0$ and it is attained by functions that are constant on the connected components of Ω. If Ω is connected, then μ_0 is simple.

We also have a max-min interpretation of the Neumann eigenvalues, by replacing $H_0^1(\Omega)$ by $H^1(\Omega)$.

Theorem 3.8 (Courant-Fischer max-min theorem)**.** *We have the following:*

- *It holds*
$$\mu_n = \inf_{u \in H_{n-1}} R(u) = \sup_{u \in E_{\lambda_n}} R(u), \tag{3.5}$$
where $H_{n-1} = H^1(\Omega) \cap E_{\mu_n}^{\perp}$.

- *It holds*
$$\mu_n = \inf_{X \in \Phi_n(H^1(\Omega))} \sup_{u \in X} R(u), \tag{3.6}$$
where $\Phi_k(H^1(\Omega)) = \{n - \textit{dimensional linear subspaces} X \subset H^1(\Omega)\}$.

Theorem 3.9 (Courant-Fischer max-min theorem)**.** *For any $n \in \mathbb{N}^*$ we have*

$$\mu_n = \max_{\substack{V \text{ subspace of } H^1(\Omega) \\ \operatorname{codim} V = n-1}} \min_{u \in V \setminus \{0\}} R(u).$$

An immediate consequence is the following.

Corollary 3.10. *For any $n \in \mathbb{N}^*$ we have*

$$\mu_k(\Omega) \leq \lambda_k(\Omega).$$

A point of attention: a domain monotonicity result like Corollary 3.6 does not hold for the Neumann eigenvalues (see Problem 3.9).

Exercices

3.6 Show that if Ω has n connected components, then $\dim E_{\mu_1} = n$ and find a basis of E_{μ_1}.

3.7 Compute the eigenvalues of Δ_N on the interval $\Omega = (0, 1) \subset \mathbb{R}$.

3.8 Compute the eigenvalues of Δ_N on the cube $\Omega = \Pi_{i=1}^d (0, 1) \subset \mathbb{R}^d$.

3.9 Find a counter-example to the domain monotonicity for Neumann eigenvalues.

Chapter 4

A direct problem

In this chapter we will study the high frequency distribution of eigenvalues of the Dirichlet Laplace operator on a bounded domain Ω. Recall that we denote by $\lambda_n(\Omega)$ and $\mu_n(\Omega)$ the n-th eigenvalue of the Dirichlet and Neumann Laplace operator, respectively, on a domain Ω.

High frequency distribution: Weyl law

We are particularly interested in the following quantity.

Definition 4.1. The eigenvalue counting functions are

$$N_D(\lambda) = \#\{\lambda_i(\Omega) \mid \lambda_i \leq \lambda\} \qquad \text{and} \qquad N_N(\lambda) = \#\{\mu_i(\Omega) \mid \mu_i \leq \lambda\}.$$

We are interested in the *Weyl law*, i.e., the asymptotic behavoir of $N_{N/D}(\lambda)$ as $\lambda \to +\infty$. We start by determining it for the case of an hypercube. This will be helpful, since the classical proof of the Weyl law for a general domain is based on a covering argument via hypercubes.

Lemma 4.2. *Let $\Omega = (0, L)^d$, $L > 0$, then*[1]

$$N_{N/D}(\lambda) \sim \lambda^{d/2} \frac{\omega_d}{\pi^d} |\Omega|.$$

Here, ω_d is the volume of the unit ball of $\mathbb{R}^d$, and $|\Omega| = L^d$ is the volume of Ω.

Remark 4.3. The above asymptotic is telling us that the high frequency distribution of the eigenvalues of the Laplace operator contains informations (at least) about

- the dimension of the ambient space,
- the volume of the domain.

Although outside the scope of these lectures, the presence of ω_d also signals a dependence on the Euclidean structure with respect to which the Laplace operator is "natural".

[1]The notation $f(\lambda) \sim g(\lambda)$ is a shorthand for $\lim_{\lambda \to +\infty} f(\lambda)/g(\lambda) = 1$.

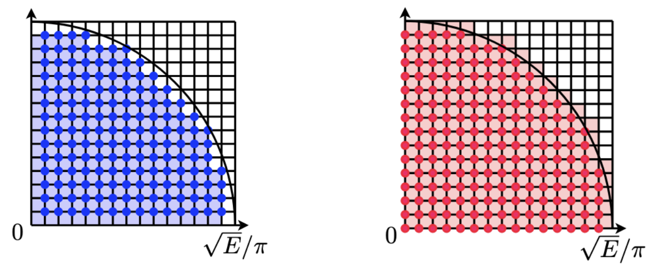

Figure 4.1: Counting the eigenvalues of Δ_D (left) and Δ_N (right) less than E on a cube of side length $L = 1$. Image from [5].

Proof. An easy adaptation of the solutions to Problems 3.3 and 3.2 shows that the eigenvalues on Ω of Δ_D and Δ_N are exactly

$$\frac{\pi^2}{L^2}\sum_{i=1}^{d} k_i^2, \quad \text{where} \quad k_i \in \begin{cases} \mathbb{N}^* & \text{for } \Delta_D \\ \mathbb{N} & \text{for } \Delta_N. \end{cases}$$

We will focus on the Dirichlet case, the Neumann one being analogous.

To determine $N_D(\lambda)$ we must count the eigenvalues less than a certain $\lambda > 0$. We compute

$$\frac{\pi^2}{L^2}\sum_{i=1}^{d} k_i^2 \le \lambda \iff \sqrt{\sum_{i=1}^{d} k_i^2} \le \frac{\sqrt{\lambda}L}{\pi} \iff |(k_1,\ldots,k_d)|_{\mathbb{R}^d} \le \frac{\sqrt{\lambda}L}{\pi}.$$

Thus, counting the eigenvalues of Δ_D less than λ amounts to count the number of point with integer and positive coordinates laying inside a disk of radius $\sqrt{\lambda}L/\pi$ (see Figure 4.1).

Let $k = (k_1,\ldots,k_d)$, $k_i \in \mathbb{N}^*$, and define $C_k = \Pi_{i=1}^d (k_i - 1, k_i)$. That is, C_k is the hypercube of unit side length such that the multi-integer k is at the upper-right angle. All hypercubes C_k with $|k|_{\mathbb{R}^d} \le \sqrt{\lambda}L/\pi$ lie in the region

$$A_\lambda = B\left(0, \frac{\sqrt{\lambda}L}{\pi}\right) \cap \{(x_1,\ldots,x_d) \mid x_i \ge 0, i = 1,\ldots,d\}.$$

It is immediate to compute that

$$|A_\lambda| = \omega_d \frac{\lambda^{d/2}L^d}{\pi^d}\frac{1}{2^d}.$$

Since $|C_k| = 1$, the above volume is an upper bound for the number of eigenvalues less than λ, that is

$$N_D(\lambda) \le \omega_d \frac{\lambda^{d/2}L^d}{(2\pi)^d}. \tag{4.1}$$

A similar reason for the Neumann case, yields the lower bound

$$N_N(\lambda) \geq \omega_d \frac{\lambda^{d/2} L^d}{(2\pi)^d}. \tag{4.2}$$

The difference between $N_D(\lambda)$ and $N_N(\lambda)$ is exactly the number of $k = (k_1, \ldots, k_d)$ such that $|k|_{\mathbb{R}^d} \leq \sqrt{\lambda} L/\pi$ such that at least one $k_i = 0$. That is,

$$N_N(\lambda) - N_D(\lambda) = \# \left\{ (k_1, \ldots, k_d) \in (\mathbb{N}^*) \mid \Pi_{i=1}^d k_i = 0,\, |k|_{\mathbb{R}^d} \leq \frac{\sqrt{\lambda} L}{\pi} \right\}.$$

Let $\ell \in \{1, \ldots, d\}$. Proceeding as above, one can upper bound the number of $k = (k_1, \ldots, k_d)$ having *exactly* ℓ non-zero components by $K\lambda^{\ell/2} L^\ell$, where $K = \max_{\ell=1,\ldots,d} \omega_\ell (2\pi)^{-\ell} \ell^{-1}$. This implies

$$N_N(\lambda) - N_D(\lambda) \leq K \sum_{i=0}^{d-1} \lambda^{\ell/2} L^\ell \leq C \left(1 + \lambda^{\frac{d-1}{2}} L^{d-1} \right).$$

Toghether with (4.1) and (4.2), this completes the proof of the statement. □

Remark 4.4. We actually proved the more precise estimate

$$N_{D/N}(\lambda) = \lambda^{d/2} \frac{\omega_d}{\pi^d} |\Omega| + O\left(|\Omega|^{\frac{d-1}{d}} \lambda^{\frac{d-1}{2}} \right),$$

where the big-O is uniform in λ and L.

It turns out that the structure of the Weyl law for the rectangle is quite universal. Independently of the shape of the domain, the asymptotic behavior has always the same structure.

Theorem 4.5 (Weyl law)**.** *Let Ω be a bounded regular domain. Then,*

$$N_{D/N}(\lambda) \sim \frac{\omega_d}{(2\pi)^d} |\Omega| \lambda^{d/2}, \qquad \text{as } \lambda \to +\infty.$$

Proof. The proof is divided in two steps: first, we assume that the domain is a union of finitely-many cubes, for which we already computed the Weyl law in Lemma 4.2. Secondly, we approximate the domain from the interior and from the exterior by unions of finitely-many cubes.

Step 1. Let $\Omega = \bigcup_{i=1}^N Q_i$ where each Q_i is an hypercube and the Q_i are almost-disjoint, in the sense they can intersect only on their boundaries.

Let $\{\lambda_n(Q_i)\}_{n,i}$ and $\{\mu_n(Q_i)\}_{n,i}$ be the collections of all the Dirichlet and Neumann eigenvalues of the hypercubes, respectively. We then order these sets in two sequences $\{\tilde\lambda_n\}_n$ and $\{\tilde\mu_n\}_n$ such that $\tilde\lambda_1 \leq \tilde\lambda_2 \leq \ldots$ and $\tilde\mu_1 \leq \tilde\mu_2 \leq \ldots$. Thanks to the disjointness of the Q_is, these eigenvalues are associated with the operators

$$\tilde\Delta_D = \bigoplus_{i=1}^N \Delta_D^{Q_i} \qquad \text{and} \qquad \tilde\Delta_N = \bigoplus_{i=1}^N \Delta_N^{Q_i}, \tag{4.3}$$

where $\Delta_{D/N}^{Q_i}$ denotes the Dirichlet/Neumann Laplace operator on the cube Q_i.

The forms associated with these operators have, respectively, domains

$$V_0 = \{u \in L^2(\Omega) \mid u|_{Q_i} \in H_0^1(Q_i) \text{ for } i = 1, \ldots, N\},$$
$$V = \{u \in L^2(\Omega) \mid u|_{Q_i} \in H^1(Q_i) \text{ for } i = 1, \ldots, N\}.$$

Observe, in particular, that we have

$$V_0 \subset H_0^1(\Omega) \subset H^1(\Omega) \subset V. \tag{4.4}$$

Moreover, as in the min-max theorems (Theorems 3.5 and 3.9) we have the characterization

$$\tilde{\lambda}_k = \inf_{X \in \Phi_k(V_0)} \sup_{u \in X} R(u) \qquad \text{and} \qquad \tilde{\mu}_k = \inf_{X \in \Phi_k(V)} \sup_{u \in X} \tilde{R}(u). \tag{4.5}$$

Here $\tilde{R}(u)$ is a modified version of the Rayleigh quotient:

$$\tilde{R}(u) = \frac{\sum_{i=1}^n \int_{Q_i} |\nabla u|^2}{\int_\Omega |u|^2}.$$

Taking into account the sets appearing in the infimum in (4.5) and in the min-max theorems (Theorems 3.5 and 3.9), and the inclusions (4.4), we easily obtain that

$$\tilde{\mu}_k \le \mu_k \le \lambda_k \le \tilde{\lambda}_k \quad \forall k \in \mathbb{N}^* \implies \tilde{N}_D(\lambda) \le N_D(\lambda) \le N_N(\lambda) \le \tilde{N}_N(\lambda).$$

Here, we denoted by $\tilde{N}_{D/N}(\cdot)$ the eigenvalue counting functions of the operators in (4.3).

Since it is immediate to observe that $\tilde{N}_{D/N}(\lambda)$ are obtained by summing the eigenvalue counting functions on all the hypercubes Q_i, the result follows by applying the Weyl law of Lemma 4.2.

Step 2. Let Ω be a bounded domain. Given any $\varepsilon > 0$ there exists Ω_- and Ω_+, finite union of rectangles, such that $\Omega_- \subset \Omega \subset \Omega_+$ and $|\Omega_+ \setminus \Omega_-| \le \varepsilon$. Domain monotonicity implies that $\lambda_k(\Omega_+) \le \lambda_k(\Omega) \le \lambda_k(\Omega_-)$ and hence

$$N_D^{\Omega_-} \le N_D(\lambda) \le N_D^{\Omega_+}(\lambda).$$

Then we have

$$\limsup_{\lambda \to +\infty} \frac{N_D(\lambda)}{\lambda^{d/2}} \le \limsup_{\lambda \to +\infty} \frac{N_D^{\Omega_+}(\lambda)}{\lambda^{d/2}} = \frac{\omega_d}{(2\pi)^d} |\Omega_+| \le \frac{\omega_d}{(2\pi)^d} (|\Omega| + \varepsilon).$$

Similarly,

$$\liminf_{\lambda \to +\infty} \frac{N_D(\lambda)}{\lambda^{d/2}} \ge \liminf_{\lambda \to +\infty} \frac{N_D^{\Omega_-}(\lambda)}{\lambda^{d/2}} = \frac{\omega_d}{(2\pi)^d} |\Omega_-| \ge \frac{\omega_d}{(2\pi)^d} (|\Omega| - \varepsilon).$$

As $\varepsilon > 0$ is arbitrary, the statement follows. □

Remark 4.6. The only time where the regularity assumption is used in the proof is to guarantee the existence of Ω_+ and Ω_- such that $|\Omega_+ \setminus \Omega_-| \leq \varepsilon$. This property is a sort of "negligeability of the boundary".

We stress however, that this regularity is essential. There exists counter-examples to the Weyl law for sets with wildly irregular boundaries (e.g., fractals).

A great deal of work has been dedicated to improve the Weyl law, and in particular to determine the next term in the expansion.

Conjecture 4.7 (Weyl conjecture)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary. Then,*

$$N_{D/N}(\lambda) = \frac{\omega_d}{(2\pi)^d}|\Omega|\lambda^{d/2} \mp \frac{1}{4}\frac{\omega_{d-1}}{(2\pi)^{d-1}}\lambda^{\frac{d-1}{2}}|\partial\Omega| + o(\lambda^{\frac{d-1}{2}}).$$

Here, the second term has negative sign for Dirichlet boundary conditions and positive for Neumann ones.

This conjecture was (almost) shown by Victor Ivrii in 1982, subject to the conjecture that the set of periodic billiards has measure zero in a bounded domain with smooth boundary.

Exercises

4.1 Show that

$$\lambda_k(\Omega) \sim \frac{(\omega_d|\Omega|k)^{2/d}}{4\pi^2}$$

Chapter 5

An inverse problem

In this chapter we will present an answer to the question "Can one hear the shape of a drum?", which is also the title of the 1966 paper by Mark Kac [4] that made it famous.

The meaning of this question is the following: Knowing the spectrum of Δ on a set Ω (which corresponds to the sound of an Ω-shaped drum), is it possible to recover its shape? As we will see, the answer is negative in general.

Isospectral domains

Definition 5.1. Two bounded open sets $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ are *isospectral* if

$$\lambda_k(\Omega_1) = \lambda_k(\Omega_2) \qquad \text{for any } k \in \mathbb{N}.$$

In order to remove trivial isospectral sets, we have to better precise the question. Indeed, there are certain transformations of a set that leave its spectrum invariant.

Definition 5.2. A map $F : \mathbb{R}^d \to \mathbb{R}^d$ is an *isometry* if

$$|F(x) - F(y)|_{\mathbb{R}^d} = |x - y|_{\mathbb{R}^d}, \qquad \text{for any } x, y \in \mathbb{R}^d.$$

In the plane $\mathbb{R}^2$ there are two types of isometries :

- Translations ($F(x) = ax + b$) and rotations ($F(x) = e^{i\theta}x$) are *orientation-preserving* isometries;
- Reflections (axial symmetries) are an *non orientation-preserving* isometries.

Proposition 5.3. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ be two bounded open sets such that $\Omega_2 = F(\Omega_1)$, for an isometry $F : \mathbb{R}^d \to \mathbb{R}^d$ (i.e., Ω_1 is isometric to Ω_2), then, Ω_1 and Ω_2 are isospectral.*

Proof. The essential fact is that Δ is invariant under isometries. Let us denote by Δ_1 and Δ_2 the Laplace operators on Ω_1 and Ω_2, respectively. These are acting on the state spaces $L^2(\Omega_1)$ and $L^2(\Omega_2)$, respectively. Consider the transformation $T : L^2(\Omega_2) \to L^2(\Omega_1)$ defined by

$$Tu(x) = u(F(x)), \qquad \text{for all } u \in L^2(\Omega_2),\ x \in \Omega_1.$$

Let us show that T is a unitary transformation (i.e., $\|Tu\|_{L^2(\Omega_1)} = \|u\|_{L^2(\Omega_2)}$). Since F is an isometry, it is not difficult to show that $DF(x)$ is an orthogonal matric, and in particular $|\det DF(x)| = 1$, for any $x \in \mathbb{R}^d$ (see Problem 5.2). Then, for any $u \in L^2(\Omega_2)$, via the change of variables $y = F(x)$, we have

$$\int_{\Omega_1} |Tu(x)|^2\,dx = \int_{\Omega_1} |u(F(x))|^2\,dx = \int_{\Omega_2} |u(y)|^2 |\det DF(y)|\,dy = \int_{\Omega_2} |u(y)|^2\,dy.$$

We now claim that $\Delta_2 = T^{-1} \circ \Delta_1 \circ T$, i.e., Δ_1 and Δ_2 are unitary equivalent. Indeed, by the chain rule,

$$\nabla[Tu](x) = \partial_{x_i}[u \circ F](x) = DF(x).\nabla u(x).$$

Then, if $u \in \mathrm{Dom}(\Delta_2)$ and $v \in H_0^1(\Omega_2)$, thanks to the fact that $DF(x)$ is orthogonal, it holds

$$\begin{aligned}
\langle T^{-1} \circ \Delta_1 \circ Tu, v\rangle &= \langle \Delta_1 \circ Tu, Tv\rangle \\
&= \int_{\Omega_2} \langle \nabla[Tu], \nabla[Tv]\rangle\,dx \\
&= \int_{\Omega_2} \langle DF(x).\nabla u(x), DF(x).\nabla v(x)\rangle\,dx \\
&= \int_{\Omega_2} \langle \nabla u(x), \nabla v(x)\rangle\,dx \\
&= \langle \Delta_2 u, v\rangle.
\end{aligned}$$

Since v is arbitrary and $H_0^1(\Omega_2)$ is dense in $L^2(\Omega_2)$, this proves the claim.

To complete the proof, it suffices to show that two unitary equivalent operators have the same spectrum. This is straightforward, indeed if φ is an eigenfunction for Δ_2 associated to the eigenvalue λ, we have that $T\varphi$ is an eigenfunction for Δ_1 for λ, as one easily verifies. □

So, the natural question to be posed is :

If Ω_1 and Ω_2 are isospectral, are they isometric?

Exercices

5.1 Show that two isospectral sets have the same volume.

5.2 Show that F is an isometry if and only if $\langle F(x), F(y)\rangle = \langle x, y\rangle$.

5.3 Show that any isometry is affine and bijective.

We cannot hear the shape of plane drum

We start by a gluing lemma that will be instrumental in the proof of the main result of this section.

Lemma 5.4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Let $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ be two open sets such that $\Omega_1 \cap \Omega_2 = \emptyset$, but such that $\Gamma := \partial\Omega_1 \cap \partial\Omega_2 \neq \emptyset$ and $\overline{\Omega_1} \cup \overline{\Omega_2} = \bar{\Omega}$.*

Let $\lambda \in \mathbb{R}$ and consider $u_1 \in C^\infty(\Omega_1)$ and $u_2 \in C^\infty(\Omega_2)$ such that

$$\Delta u_1 = \lambda u_1 \text{ on } \Omega_1 \qquad \text{and} \qquad \Delta u_2 = \lambda u_2 \text{ on } \Omega_2. \tag{5.1}$$

Observe that by continuity, we can extend both u_1 and u_2 to the common boundary Γ. We define the function $u : \Omega \to \mathbb{R}$ by

$$u(x) = \begin{cases} u_1(x) & \text{if } x \in \Omega_1 \cup \Gamma, \\ u_2(x) & \text{if } x \in \Omega_2. \end{cases}$$

Then, the following are equivalent:

a. *on Γ it holds that $u_1 = u_2$ and $\frac{\partial u_1}{\partial \vec{\nu}} = \frac{\partial u_2}{\partial \vec{\nu}}$;*

b. *$u \in C^\infty(\Omega)$ and $\Delta u = \lambda u$ on Ω.*

Proof. The fact that *b.* implies *a.* is straightforward. We now provide an argument for the opposite implication.

Let us denote by $\vec{\nu}_1$ and $\vec{\nu}_2$ the outward pointing normals to $\partial\Omega_1$ and $\partial\Omega_2$, respectively. Let $\varphi \in C_c^\infty(\Omega)$, by the Green formula (with boundary terms), and denoting with $\langle \cdot, \cdot \rangle$ the distributional duality, we have

$$\begin{aligned} -\langle \Delta u, \varphi \rangle &= \int_\Omega u \, \Delta\varphi \\ &= \int_{\Omega_1} u_1 \, \Delta\varphi + \int_{\Omega_2} u_2 \, \Delta\varphi \\ &= \int_{\Omega_1} \Delta u_1 \, \varphi + \int_{\partial\Omega_1} u_1 \frac{\partial\varphi}{\partial\vec{\nu}_1} - \int_{\partial\Omega_1} \frac{\partial u_1}{\partial\vec{\nu}_1}\varphi \\ &\quad + \int_{\Omega_2} \Delta u_2 \, \varphi + \int_{\partial\Omega_2} u_2 \frac{\partial\varphi}{\partial\vec{\nu}_2} - \int_{\partial\Omega_2} \frac{\partial u_2}{\partial\vec{\nu}_2}\varphi. \end{aligned}$$

Observe that $\vec{\nu}_1 = -\vec{\nu}_2$ on Γ, and that φ and $\frac{\partial\varphi}{\partial\vec{\nu}_1}$ vanish on $\partial\Omega$. This implies that in the above the integrals on $\partial\Omega_1$ and $\partial\Omega_2$ are in fact integrals on Γ and that $\frac{\partial\varphi}{\partial\vec{\nu}_1} = -\frac{\partial\varphi}{\partial\vec{\nu}_2}$. Thus, using (5.1) we have

$$\langle \Delta u, \varphi \rangle = \langle \lambda u, \varphi \rangle, \qquad \text{for any } \varphi \in C_c^\infty(\Omega).$$

That is, $\Delta u = \lambda u$ in the sense of distributions.

By assumption *a.*, we immediately deduce that $u \in C^1(\Omega)$ which, by classical elliptic regularity results, yields that $u \in C^\infty(\Omega)$ and $\Delta u = \lambda u$ in the classical sense. □

We are in a position to answer negatively to the main question of this chapter. The proof we present here is taken from [2].

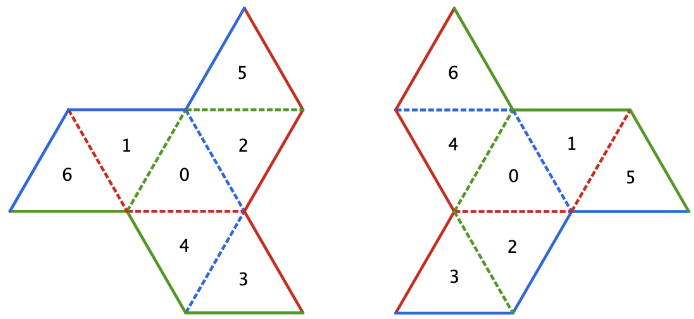

Figure 5.1: The starting point: two isometric propellers. Image from [3].

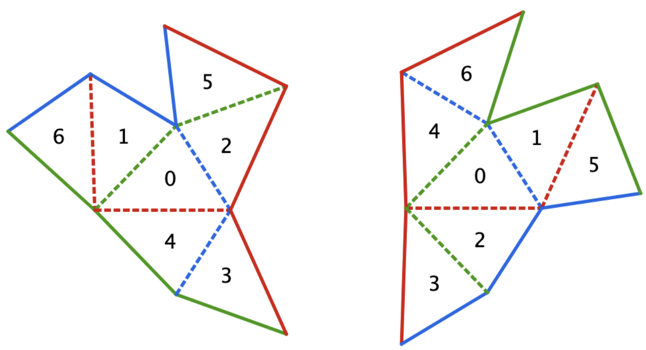

Figure 5.2: The counterexample: two non-isometric sets. Image from [3].

Theorem 5.5. *In the plane $\mathbb{R}^2$ there exist isospectral domains that are not isometric.*

Proof. To prove the theorem, we will construct two isospectral sets that are not isometric, via a technique known as transplantation. We start by consideing the two helicoidal domains of Figure 5.1. These are composed of 7 equilateral triangles, that we have aribtrarily labelled from **0** to **6**. The boundary of the domains is traced in a continuous line, while the dashed lines denotes the common boundaries of the triangles. These two domains are isometric, since they can be obtained one from another via a $\pi/2$ clockwise rotation.

We thus modify the propellers in Figure 5.1 by changing the central triangles (the ones labelled **0**) to make them non-equilateral (but still equal), and then changing all the other triangles in order to preserve the symmetries by reflection between triangles that share a boundary, as in Figure 5.2. Since the only isometry that sends the left central triangles to the right central one is a translation, it is follows that the two domains are no longer symmetric.

Let λ be an eigenvalue for the domain on the left, with associated eigenfunction φ. We aim to show that λ is also an eigenvalue for the domain on the right by explicitly constructing its eigenfunction as a modification of φ. Since this procedure is perfectly symmetric, this suffices to show that two domains are indeed isospectral.

We denote by $\varphi_0, \dots, \varphi_6$ the restrictions of φ to each of the triangles of the domain on the left. Let us also introduce the isometries $\tau_{i,j}$, for $0 \le i, j \le 6$, that sends the i-th triangle on the left to the j-th triangle on the right (preserving the colors of each side, i.e., the labels of the neighbouring triangles). Being isometries, we have that $\Delta(\varphi_i \circ \tau_{i,j}) = \lambda \varphi_i \circ \tau_{i,j}$. If we fix for any $j \in \{0, \dots, 6\}$ a set $A_j \subset \{0, \dots, 6\}$, and define the function ψ on the domain at the right, whose restriction ψ_j to the j-th triangle is

$$\psi_j = \sum_{i \in A_j} c_i \, \varphi_i \circ \tau_{i,j}, \qquad c_i \in \mathbb{R}, \tag{5.2}$$

we then have $\Delta\psi_j = \lambda\psi_j$ for all $j \in \{0, \dots, 6\}$. The problem is at the boundaries of the domain (the filled lines) and of the triangles (the dashed lines): In order to be an eigenfunction, the function ψ must be C^∞ at these boundaries and vanish on the boundaries of the domain.

We will now show how to choose the linear combinations (5.2) in order to satisfy these boundary conditions. We start by letting $A_0 = \{1, 2, 4\}$ and the constants equal to 1, that is by choosing the restriction of ψ to the central triangle **0** to be

$$\psi_0 = \varphi_1 \circ \tau_{0,1} + \varphi_2 \circ \tau_{0,2} + \varphi_4 \circ \tau_{0,4}.$$

We now look at how can we define ψ on the triangle **4** in order for ψ_0 and ψ_4 to glue correctly. To do this we have to analyse how the φ_i are connected one to the other, and recall that the boundary between **0** and **4** is the green one:

- φ_1 continues along the green boundary towards φ_0;
- φ_2 continues along the green boundary towards φ_5;
- φ_4 vanishes along the green boundary (since it is the boundary of the full domain).

Thus, in order to be continuous along the green boundary between **0** and **4**, we have to take $\psi_4 = \varphi_0 \circ \tau_{4,0} + \varphi_5 \circ \tau_{4,5} + g$, where g vanishes on the green boundary.

We now look at the red boundary Π of **4**, which is part of the boundary of the domain. Here we have that $\psi_4|_\Pi = \varphi_0 \circ \tau_{4,0} + g$, since φ_5 vanishes on the red boundary. Since φ_0 is connected to φ_4 along the dashed red boundary, to have $\psi_4|_\Pi = 0$ we must choose $g = -\varphi_4 \circ \tau_{4,4}$. Observe that this is coherent, since φ_4 vanishes on the green boundary. To sum up, we have

$$\psi_4 = \varphi_0 \circ \tau_{4,0} + \varphi_5 \circ \tau_{5,2} - \varphi_4 \circ \tau_{4,4},$$

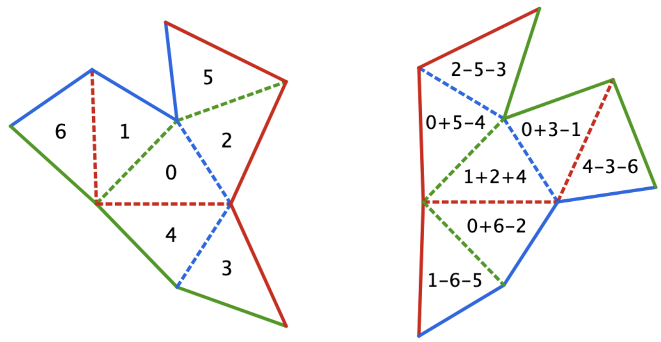

Figure 5.3: An appropriate linear combination.

and with this choice we have that the gluing of ψ_0 and ψ_4 is C^1 along the common (green) side between **0** and **4**, and ψ_4 vanishes at the boundary of the domain (red side of **4**).

Observe now that by applying Lemma 5.4, we can guarantee that the gluing u between ψ_0 and ψ_4 is indeed C^∞, such that $\Delta u = \lambda u$ on the union of the two triangles, and that vanishes on the boundary of the domain.

One can then proceed in this fashion to obtain all other ψ_js. The procedure is mechanical and forced by the choices of labelling and coloring of the sides. The resulting combinations are depicted in Figure 5.3. This then shows that if φ is an eigenfunction of Δ on the domain to the left, the function ψ that we obtain is an eigenfunction on the domain to the right, associated with the same eigenvalue. We have thus completed the proof. □

Generalizations

The above proof, although correct, is not completely satisfactory since it is based on the "guessing" of the correct two sets from which to start with. One is naturally left with the question of how the propellers of Figure 5.1 have been found and why do they work. The main tool used to derive these sets is the following theorem that gives purely geometrical conditions for manifolds to be isospectral. (On a Riemannian manifolds, the role of the Laplace operator is played by the Laplace-Beltrami operator.)

Theorem 5.6 (Sunada [8], Pesce [6])**.** *Let G be a finite group acting freely*[1] *on a manifold M, and H and H' be subgroups. Then, the following are equivalent:*

- *for every conjugacy class C of G, we have $\#(H \cap C) = \#(H' \cap C)$;*
- *for every G-invariant Riemannian metric on M, the quotient Riemannian manifolds $H \setminus M$ and $H' \setminus M$ have the same Laplace-Beltrami spectrum.*

[1] That is, every non-trivial element of G acts without fixed points on M.

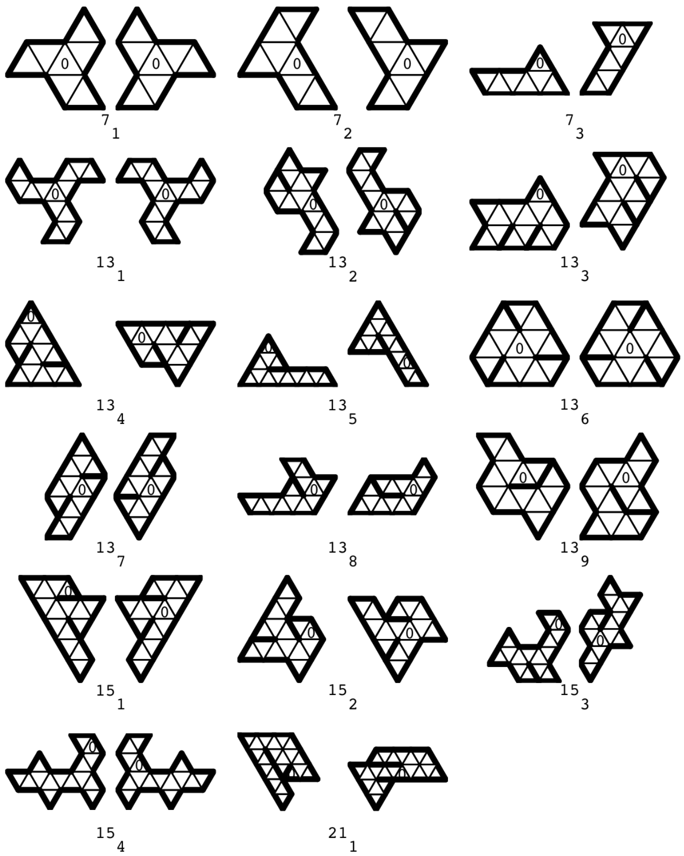

Figure 5.4: Some couples of sets which can be used to obtained isospectral sets. As in the proof of Theorem 5.5 each couple yields a continuum of counterexamples, depending on the chosen deformation of the triangle labelled **0**. Image from [2].

Using this theorem, it is possible to find many planar sets on which the transplantation technique we used in the proof of Theorem 5.5 can be applied. We refer to [2] for a more detailed description of this procedure, thanks to which Figures 5.4 and 5.5 have been obtained.

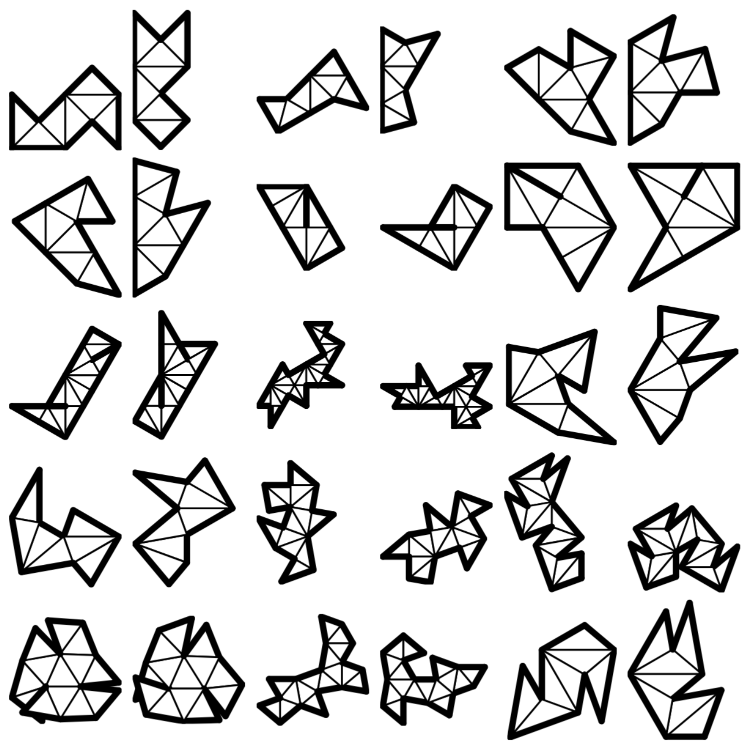

Figure 5.5: Some special examples of isospectral sets, obtained by deformation of the sets in Figure 5.4. Image from [2].

Hints

2.1. Consider any compactly supported function $\varphi \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}\varphi \subset (0,1)$, and define $\varphi_n(x) = \varphi(nx)$. Show that $\|\Delta\varphi_n\|_{L^2((0,1))}/\|\varphi_n\|_{L^2((0,1))} \to +\infty$ as $n \to +\infty$.

3.1. Use the Riesz representation theorem.

3.2. We need to find the λs such that $u'' = \lambda u$ admits a solution such that $u(0) = u(1) = 0$.

3.3. Look for solutions in the form $u(x_1, \dots, x_d) = u_1(x_1) \cdots u_d(x_d)$.

3.4. Use the min-max principle.

3.5. Use (again) the min-max principle.

3.7. We need to find the λs such that $u'' = \lambda u$ admits a solution such that $u'(0) = u'(1) = 0$.

3.8. Look for solutions in the form $u(x_1, \dots, x_d) = u_1(x_1) \cdots u_d(x_d)$.

3.9. Exploit the solution to Problem 3.8.

5.1. Use Weyl law.

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